

# Invariant measure and long time behavior of regular solutions of the Benjamin-Ono equation

Mouhamadou SY

*Université de Cergy-Pontoise  
Laboratoire AGM UMR 8088 CNRS  
2 av. Adolphe Chauvin, 95302 Cergy-Pontoise Cedex, France  
mouhamadou.sy@u-cergy.fr*

## Abstract

The Benjamin-Ono equation describes the propagation of internal waves in a stratified fluid. In the present work, we study large time dynamics of its regular solutions via some probabilistic point of view. We prove the existence of an invariant measure concentrated on  $C^\infty(\mathbb{T})$  and establish some qualitative properties of this measure. We then deduce a recurrence property of regular solutions. The approach used in this paper applies to other equations with infinitely many conservation laws, such as the KdV and cubic Schrödinger equations in 1D.

**Keywords:** Benjamin-Ono equation, invariant measure, long time behavior, regular solutions, inviscid limit

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
1.1	The problem and statement of the main result . . . . .	2
1.2	Notations, spaces and forces . . . . .	4
<b>2</b>	<b>Preliminaries</b>	<b>5</b>
2.1	Conservation laws . . . . .	5
2.2	Deterministic estimates . . . . .	6
2.3	Probabilistic estimates . . . . .	8
<b>3</b>	<b>Wellposedness of the stochastic BOB equation</b>	<b>10</b>
<b>4</b>	<b>Invariant measures for the viscous problem</b>	<b>14</b>
<b>5</b>	<b>Invariant measure for the BO equation</b>	<b>16</b>
5.1	Some convergence results of stochastic BOB to BO . . . . .	16
5.2	An accumulation point for the viscous measures . . . . .	17
<b>6</b>	<b>Qualitative properties of the measure</b>	<b>20</b>
6.1	Absolute continuity of some observables w.r.t. to the Lebesgue measure	20
6.2	About the dimension of the measure $\mu$ . . . . .	23
6.3	A Gaussian decay property for the measure $\mu$ . . . . .	27

<b>A Proof of Lemma 3.3</b>	<b>28</b>
<b>B The periodic Hilbert transform</b>	<b>30</b>

## 1 Introduction

### 1.1 The problem and statement of the main result

The Benjamin-Ono (BO) equation

$$\partial_t u + H \partial_x^2 u + u \partial_x u = 0 \quad (1.1)$$

describes the propagation of internal waves in a stratified fluid. The operator  $H$  entering the equation is the Hilbert transform, which can be defined in Fourier setting as the multiplier by  $-i \operatorname{sgn}$  (see Appendix). We assume that  $u(t, x)$  is a real-valued function,  $t \in \mathbb{R}_+$  and  $x$  belongs to the torus  $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ . In this setting, existence and uniqueness of solution hold in  $L^2(\mathbb{T})$  (see [Mol08, MP12]). In our work we need the wellposedness of the problem only in Sobolev spaces  $H^s(\mathbb{T})$  with  $s \geq 2$ , so we refer the reader to [ABFS89] where wellposedness was proved for  $s > \frac{3}{2}$ .

Positioning in  $L^2 := L^2(\mathbb{T})$ , the wellposedness of (1.1) generates a topological dynamical system (DS)  $(L^2, \phi_t)$ , where  $\phi_t$  is the flow of the BO equation (1.1). One of the great questions of the qualitative study of evolution PDE is to describe the long time behavior of such DS.

Given a Borel measure  $\mu$  on  $L^2$ , we say that  $\mu$  is invariant for the DS  $(L^2, \phi_t)$  if for any Borel set  $A$  of  $L^2$ , we have

$$\mu(\phi_t^{-1}A) = \mu(A), \quad \forall t. \quad (1.2)$$

When such a measure exists, the triple  $(L^2, \phi_t, \mu)$  is called measurable dynamical system (MDS). If in addition  $\mu$  is finite, the Poincaré recurrence theorem implies that the dynamics is recurrent, that is,  $\mu$ -almost every orbit goes back in any neighborhood of its origin in finite time. The Von Neumann and Birkhoff ergodic theorems can also be applied to give further informations on the long time behavior of the system.

Matsuno [Mat84] derived (at least formally) infinitely many conservation laws for the BO equation (1.1), they have the form

$$E_n(u) = \|u\|_n^2 + R_n(u), \quad n \in \frac{\mathbb{N}}{2}, \quad (1.3)$$

where  $\|\cdot\|_n$  stand for the homogenous Sobolev norm of order  $n$  and  $R_n$  is a lower order term.

In [TV13, TV14, TV15, Den15, DTV15] the authors construct a sequence of invariant Gaussian type measures  $\{\mu_n\}$  for  $(L^2, \phi_t)$  satisfying the following:

- $\mu_n$  is concentrated on  $H^s(\mathbb{T})$ , for  $s < n - \frac{1}{2}$ , (\*)
- $\mu_n(H^{n-1/2}(\mathbb{T})) = 0$ . (\*\*)

Formally,  $\mu_n$  is defined as a renormalization of

$$d\mu_n(u) = e^{-E_n(u)} du = e^{-R_n(u)} e^{-\|u\|_n^2} du, \quad (1.4)$$

where  $E_n(u)$  and  $R_n(u)$  are the quantities given in (1.3). The authors construct a Gaussian renormalization of the expression  $e^{-\|u\|_n^2} du$  on the concerned spaces and prove that  $e^{-R_n(u)}$  is an integrable density. In view of these results, a MDS for (1.1) exists in any Sobolev space and then its large time dynamics is described in the sense of the theorems mentioned above. However, these results do not apply to infinitely smooth solutions, indeed by (\*\*) we have

$$\mu_n(C^\infty(\mathbb{T})) = 0, \quad \text{for all } n. \quad (1.5)$$

In the present work we construct a measurable dynamical system for BO (1.1) on  $C^\infty(\mathbb{T})$ . Of course the Dirac measure at 0 is not the desired measure although it is invariant under the flow of BO, because it gives no information. More generally, to get substantial information on the system we have to avoid too singular measures. Another example of such measures is the one concentrated on a stationary solution. Notice that the measures  $\mu_n$  discussed above verify the following "consistency" property: every set of full  $\mu_n$ -measure is dense in  $\dot{H}^{(n-1/2)^-}$ . However, an obstruction to the construction of an invariant Gaussian type measure in  $C^\infty(\mathbb{T})$  is the non-existence of conservation law compatible with the  $C^\infty$ -regularity. In particular, the approach used in the construction of the measures  $\mu_n$  does not seem to apply.

Another method allowing to construct invariant measures (a priori not of Gaussian type) for PDE was introduced in [Kuk04, KS04] in the context of Euler and Schrödinger equations. It is based on a fluctuation-dissipation argument and consists in adding to the equation appropriately normalized damping and stochastic terms, constructing an invariant measure for the resulting problem, and passing to the limit. The idea in this work is to develop this approach in the context of the BO equation, and combine it with the structure of the conservation laws to construct a non-trivial invariant measure concentrated on  $C^\infty(\mathbb{T})$ .

We first consider the stochastic-diffusion problem (also called stochastic Benjamin-Ono-Burgers (BOB) equation)

$$\partial_t u + H \partial_x^2 u + u \partial_x u = \alpha \partial_x^2 u + \sqrt{\alpha} \eta, \quad t > 0, \quad x \in \mathbb{T}, \quad (1.6)$$

where  $\eta$  is a stochastic force and  $\alpha \in (0, 1)$  is a parameter. Then our problem (1.1) is the limit as  $\alpha \rightarrow 0$  of the stochastic-diffusion equation (1.6). After proving well-posedness in some probabilistic sense of the initial value problem of (1.6), we prove in Section 4 the existence of invariant measures for the latter and that for sufficiently regular in space noise, any invariant measure of (1.6) is concentrated on  $C^\infty(\mathbb{T})$ . Passing to the limit as the viscosity goes to zero, we prove in Theorems 5.3, 6.1 and 6.3 the following result

**Theorem 1.1.** *There is a probability measure  $\mu$  invariant under the flow of the BO equation (1.1) defined on  $H^3(\mathbb{T})$  and concentrated on the infinitely smooth functions, that is*

$$\mu(C^\infty(\mathbb{T})) = 1.$$

*This measure  $\mu$  satisfies the following properties:*

1. *For any integer  $n$  and any real number  $p \geq 1$ , we have*

$$0 < \int_{H^3} \|u\|_n^2 \mu(du) < \infty, \quad 0 < \int_{H^3} \|u\|^p \mu(du) < \infty.$$

2. There is an infinite sequence of independent preserved quantities whose laws under  $\mu$  are absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}$ .
3. The measure  $\mu$  is of at least two-dimensional nature in the sense that any compact set of Hausdorff dimension smaller than 2 has  $\mu$ -measure 0.

In our context, the Poincaré recurrence theorem will translate into the following result

**Corollary 1.2.** *For  $\mu$ -almost all  $w$  in  $C^\infty(\mathbb{T})$ , there is a sequence  $\{t_k\}$  increasing to infinity such that*

$$\lim_{k \rightarrow \infty} \|S_{t_k} w - w\|_n = 0, \quad \forall n. \quad (1.7)$$

Here  $S_t$  denotes the flow of the Benjamin-Ono equation (1.1) on  $H^3(\mathbb{T})$ .

Roughly speaking, it states that for sufficiently large time the regular solutions to the BO equation (1.1) go back near their initial states.

In the construction of such a measure, we need the increasing Sobolev regularity provided by the infinite sequence of conservation laws. It is the question of the sequence  $\{\|u\|_n^2\}$  which comes from the dispersive term  $H\partial_x^2 u$  of the BO equation (1.1).

The KdV and cubic 1D NLS equations have infinitely many conservation laws whose form is similar to (1.3), this makes our approach to apply for these equations. Recall that an infinite sequence of invariant Gaussian type measures of increasing regularity was constructed for KdV and cubic 1D NLS in [Zhi01a, Zhi01b]. However, the nonlocal nature of the BO equation makes it more complicated than KdV and cubic 1D NLS which are better understood.

Let us discuss briefly an equation having infinitely many conservation laws but not including an increasing Sobolev regularity. Consider the non-viscous Burgers equation

$$\partial_t u + u \partial_x u = 0. \quad (1.8)$$

It is easy to check that an infinite sequence of conservation laws is given by the quantities

$$L_p(u) = \int u^p, \quad p \geq 1. \quad (1.9)$$

Our approach does not apply to (1.8) because of the lack of dispersion.

In [Sy16], the fluctuation-dissipation approach is used to construct a non-trivial invariant measure concentrated on  $H^2 \times H^1$  for the cubic Klein-Gordon equation

$$\partial_t^2 u - \Delta u + u + u^3 = 0, \quad (1.10)$$

considered on a bounded domain  $D \subset \mathbb{R}^3$  or on the torus  $\mathbb{T}^3$ .

## 1.2 Notations, spaces and forces

Let  $A$  and  $B$  be two positive quantities, we write

$$A \lesssim B \quad (1.11)$$

if there is a universal constant  $\lambda \geq 0$  such that  $A \leq \lambda B$ .

For a real number  $r$ ,  $r^+$  (resp.  $r^-$ ) denotes  $r + \varepsilon$  (resp.  $r - \varepsilon$ ) where  $\varepsilon$  is a positive number close enough to 0.

$\mathbb{Z}_0$  denotes the set of nonzero integers.

$\dot{H}(\mathbb{T}) = \{u \in L^2(\mathbb{T}) : \int_{\mathbb{T}} u(x) dx = 0\}$ .

$\dot{H}^s(\mathbb{T}) = \{u \in \dot{H}(\mathbb{T}) : D^s u \in \dot{H}(\mathbb{T})\}$ ,  $D^s$  is the  $s$ th derivative of  $u$ , where  $s \geq 0$ .

The  $\dot{H}^s$ -norm is denoted by  $\|\cdot\|_s$  when  $s > 0$  and the  $L^2$ -norm is denoted by  $\|\cdot\|$ .

For a functional  $A(u)$ ,  $\partial_u A = A'$  and  $\partial_u^2 A = A''$  denote the first and second derivative of  $A$  w.r.t.  $u$ .

The sequence  $\{e_n, n \in \mathbb{Z}_0\}$  is defined by

$$e_n(x) = \begin{cases} \frac{\sin(nx)}{\sqrt{\pi}}, & \text{for } n > 0, \\ \frac{\cos(nx)}{\sqrt{\pi}}, & \text{for } n < 0. \end{cases}$$

and forms an orthonormal basis of  $\dot{H}(\mathbb{T})$ .

$(\Omega, \mathcal{F}, \mathbb{P})$  is a complete probability space.  $\mathcal{F}_t$  is a right-continuous filtration augmented w.r.t.  $(\mathcal{F}, \mathbb{P})$ . Given numbers  $\{\lambda_n\} \subset \mathbb{R}$  and a sequence of independent real standard Brownian motions  $\{\beta_n(t)\}$  adapted to  $\mathcal{F}_t$ , we define

$$\zeta(t, x) = \sum_{n \in \mathbb{Z}_0} \lambda_n \beta_n(t) e_n(x), \quad (1.12)$$

$$\eta(t, x) = \frac{d}{dt} \zeta(t, x), \quad (1.13)$$

and

$$A_s = \sum_{n \in \mathbb{Z}_0} \lambda_n^2 n^{2s}. \quad (1.14)$$

## 2 Preliminaries

### 2.1 Conservation laws

For an enough smooth function  $u$ , set

$$\begin{aligned} \mathcal{P}_1 &= \{\partial_x^\alpha u, \partial_x^\alpha H u \mid \alpha \in \mathbb{N}\}, \\ \mathcal{P}_2 &= \{(\partial_x^{\alpha_1} Z_1 u)(\partial_x^{\alpha_2} Z_2 u) \mid \alpha_i \in \mathbb{N}, Z_i \in \{Id, H\}\}. \end{aligned}$$

Generically we define  $\mathcal{P}_n$ ,  $n \geq 3$  as the set of functions of the form

$$p_n(u) = \prod_{i=1}^k Z_i(p_{j_i}(u)), \quad Z_i \in \{Id, H\}, \quad \sum_1^k j_i = n, \quad p_{j_i} \in \mathcal{P}_{j_i}. \quad (2.1)$$

To a function  $p_n(u)$  of the form (2.1), we associate

$$\tilde{p}_n(u) = \prod_{i=1}^k p_{j_i}(u), \quad (2.2)$$

and we define the quantities

$$\begin{aligned} S(p(u)) &= \sum_{i=1}^n \alpha_i, \\ M(p(u)) &= \max_{1 \leq i \leq n} \alpha_i. \end{aligned}$$

In the present work we only need the conservation laws of integer order. We describe here the corresponding remainder terms following [TV14]:

$$R_n(u) = \sum_{\substack{p(u) \in \mathcal{P}_3 \\ \tilde{p}(u) = u \partial_x^{n-1} u \partial_x^n u}} c_n(p) \int p(u) + \sum_{\substack{p(u) \in \mathcal{P}_j \ j=3, \dots, 2n+2 \\ S(p(u))=2n-j+2 \\ M(p(u)) \leq n-1}} c_n(p) \int p(u), \quad (2.3)$$

where  $c_n(p)$  are some constants. The first three conservation laws of integer order are

$$\begin{aligned} E_0(u) &= \int u^2, \\ E_1(u) &= \int (\partial_x u)^2 + \frac{3}{4} \int u^2 H \partial_x u + \frac{1}{8} \int u^4, \\ E_2(u) &= \int (\partial_x^2 u)^2 - \frac{5}{4} \int ((\partial_x u)^2 H \partial_x u + 2 \partial_x^2 u H \partial_x u) \\ &\quad + \frac{5}{16} \int (5u^2 (\partial_x u)^2 + u^2 (H \partial_x u)^2 + 2uH(\partial_x u)H(u \partial_x u)) \\ &\quad + \int \left( \frac{5}{32} u^4 H(\partial_x u) + \frac{5}{24} u^3 H(u \partial_x u) \right) + \frac{1}{48} \int u^6. \end{aligned}$$

## 2.2 Deterministic estimates

**Lemma 2.1.** *There are  $c_n^-, c_n^+ > 0$  such that for all  $u$  in  $H^n(\mathbb{T})$*

$$\frac{1}{2} \|u\|_n^2 - c_n^- \|u\|^{2n+2} \leq E_n(u) \leq 2 \|u\|_n^2 + c_n^+ \|u\|^{2n+2}. \quad (2.4)$$

**Lemma 2.2.** *For all  $\varepsilon > 0$ , there is  $C_\varepsilon > 0$  such that for all  $u$  in  $H^{n+1}(\mathbb{T})$*

$$\partial_u E_n(u, \partial_x^2 u) \leq (-2 + \varepsilon) \|u\|_{n+1}^2 + C_\varepsilon \|u\| (1 + \|u\|)^{b_n}, \quad (2.5)$$

where  $b_n$  depends only on  $n$ .

*Remark 2.3.* Since the  $L^2$ -norm is preserved by (1.1) we can deduce from (2.4) and the arguments of the proof of Lemma 2.2, by adding appropriate polynomials of  $\|u\|$ , new conservation laws  $E_n^*(u)$  and  $\tilde{E}_n(u)$  satisfying

$$0 \leq \|u\|_n^2 \leq E_n^*(u), \quad (2.6)$$

$$0 \leq \|u\|_n^2 \leq \partial_u \tilde{E}_n(u, u). \quad (2.7)$$

Inequalities (2.4) can be established using similar arguments as in the proof of Lemma 2.2.

*Proof of Lemma 2.2.* Taking into account of the properties of the Hilbert transform such as continuity in  $H^s$  and  $L^p$  ( $s \geq 0$ ,  $p \in ]1, \infty[$ ), we can "remove"  $H$  and consider the functions

$$R_n^1(u) = \int u \partial_x^{n-1} u \partial_x^n u, \quad (2.8)$$

$$R_n^{2,j}(u) = \int \prod_{i=1}^j \partial_x^{\alpha_i} u, \quad j = 3, \dots, 2n+2. \quad (2.9)$$

Here  $R_n^1(u)$  represents the first term of (2.3) and the second term of (2.3) can be estimated considering the quantities  $R_n^{2,j}(u)$ . Set

$$R_n^0 = \|u\|_n^2. \quad (2.10)$$

**Estimates concerning  $R_n^0$ :**

$$\partial_u R_n^0(u, \partial_x^2 u) = -2\|u\|_{n+1}^2 \quad (2.11)$$

**Estimates concerning  $R_n^1$ :**

$$\begin{aligned} \partial_u R_n^1(u, \partial_x^2 u) &= \int \partial_x^2 u \partial_x^{n-1} u \partial_x^n u + \int u \partial_x^{n+1} u \partial_x^n u + \int u \partial_x^{n-1} u \partial_x^{n+2} u \\ &= \int \partial_x^2 u \partial_x^{n-1} u \partial_x^n u - \int \partial_x u \partial_x^{n-1} u \partial_x^{n+1} u \\ &= (1) + (2). \end{aligned}$$

Now interpolation inequalities imply

$$|(1)|, |(2)| \leq C_1 \|u\|_{n+1}^{d_1} \|u\|^{3-d_1} \quad (2.12)$$

where

$$d_1 = \frac{4n+3}{2(n+1)} < 2.$$

Then for suitable  $b_1$

$$|\partial_u R_n^1(u, \partial_x^2 u)| \leq \varepsilon \|u\|_{n+1}^2 + C_\varepsilon^1 \|u\|^{b_1}. \quad (2.13)$$

**Estimates concerning  $R_n^{2,j}$ :**

$$\partial_u R_n^{2,j}(u, \partial_x^2 u) = \int \prod_{i=1}^j \partial_x^{\alpha_i} u \quad j = 3, \dots, 2n+2, \quad (2.14)$$

where  $\sum_{i=1}^j \alpha_i = 2n-j+4$  and  $\max_{1 \leq i \leq j} \alpha_i \leq n+1$ .

We follow two complementary cases:

- Case 1:  $\max_{1 \leq i \leq j} \alpha_i \leq n$ . Let  $(\gamma_i)$  be  $j$  real numbers such that  $\sum_{i=1}^j \frac{1}{\gamma_i} = 1$ . Then the generalized Hölder formula combined with usual interpolation inequalities shows:

$$|\partial_u R_n^{2,j}(u, \partial_x^2 u)| \leq C \prod_{i=1}^j \|u\|_{\lambda_i}, \quad (2.15)$$

where  $\lambda_i = \frac{1}{2} - \frac{1}{\gamma_i} + \alpha_i$ . Then

$$|\partial_u R_n^{2,j}(u, \partial_x^2 u)| \leq C \prod_{i=1}^j \|u\|^{\frac{n+1-\lambda_i}{n+1}} \|u\|^{\frac{\lambda_i}{n+1}}.$$

Remark now that

$$\sum_{i=1}^j \lambda_i = \sum_{i=1}^j \left( \frac{1}{2} - \frac{1}{\gamma_i} + \alpha_i \right) = 2n+4 - \frac{j}{2},$$

then

$$\sum_{i=1}^j \frac{\lambda_i}{n+1} = \frac{2n+4-\frac{j}{2}}{n+1} < 2.$$

Thus for suitable  $b_2$ ,

$$|\partial_u R_n^{2,j}(u, \partial_x^2 u)| \leq \varepsilon \|u\|_{n+1}^2 + C_\varepsilon^2 \|u\|^{b_2}. \quad (2.16)$$

- Case 2 :  $\alpha_1 = n + 1$ . Then  $\sum_{i=2}^j \alpha_i = n - j + 3 \leq n$ . We have then

$$|\partial_u R_n^{2,j}(u, \partial_x^2 u)| \leq \|u\|_{n+1} \left( \int \prod_2^j |\partial_x^{\alpha_i} u|^2 \right)^{\frac{1}{2}}. \quad (2.17)$$

Take again  $(\gamma_i)$  such that  $\sum_{i=2}^j \frac{1}{\gamma_i} = 1$ . Then

$$\begin{aligned} |\partial_u R_n^{2,j}(u, \partial_x^2 u)| &\leq \|u\|_{n+1} \prod_2^j \|\partial_x^{\alpha_i} u\|_{L^{2\gamma_i}} \\ &\leq \|u\|_{n+1} \prod_2^j \|u\|_{\lambda_i}, \quad \lambda_i = \frac{1}{2} - \frac{1}{2\gamma_i} + \alpha_i, \\ &\leq \|u\|_{n+1} \prod_2^j \|u\|^{\frac{n+1-\lambda_i}{n+1}} \|u\|^{\frac{\lambda_i}{n+1}}. \end{aligned}$$

Since  $\sum_{i=2}^j \lambda_i = n + 2 - \frac{j}{2} \leq n + \frac{1}{2}$ , we have  $\frac{1}{n+1} \sum_{i=2}^j \lambda_i < 1$  and the existence of a suitable  $b_3$  such that

$$|\partial_u R_n^{2,j}(u, \partial_x^2 u)| \leq \varepsilon \|u\|_{n+1}^2 + C_\varepsilon^3 \|u\|^{b_3}. \quad (2.18)$$

Combining (2.11), (2.13) and (2.18) with good choice of  $\varepsilon$ , we have the claim.  $\square$

### 2.3 Probabilistic estimates

Consider the initial value problem concerning the stochastic BOB equation (1.6)

$$\begin{cases} \partial_t u + H \partial_x^2 u + u \partial_x u = \alpha \partial_x^2 u + \sqrt{\alpha} \eta & t > 0, \\ u|_{t=0} = u_0. \end{cases} \quad (2.19)$$

Wellposedness of (2.19) in some probabilistic sense will be established in Proposition 3.1. Recall the definition of the following constants

$$A_s = \sum_{m \in \mathbb{Z}_0} m^{2s} \lambda_m^2,$$

which "measure" the regularity in space of the noise.

**Theorem 2.4.** *Let  $n \geq 2$ . Suppose  $A_n$  finite. There are  $\theta_n > 0$ ,  $\gamma_n > 0$  such that for any random function  $u$  in  $H^n$  solution of (2.19) issued to  $u_0$  which satisfies  $\mathbb{E} E_n(u_0) < \infty$ , we have*

$$\mathbb{E} E_n(u) + \alpha \int_0^t \mathbb{E} \|u\|_{n+1}^2 ds \leq \mathbb{E} E_n(u_0) + \alpha A_n \left[ t + c_n \int_0^t \mathbb{E} \|u\|_n^2 ds + \gamma_n \int_0^t \mathbb{E} \|u\| (1 + \|u\|)^{\theta_n} ds \right], \quad (2.20)$$

where  $c_n$  depends only on  $n$ .

*Proof.* Applying the Ito formula (see Theorem A.7.5 and Corollary A.7.6 of [KS12] for the Ito formula in Hilbert spaces) to the conservation law  $E_n(u)$ , we find



$$dE_n(u) = \partial_u E_n(u, du) + \frac{\alpha}{2} \sum_{m \in \mathbb{Z}_0} \lambda_m^2 \partial_u^2 E(u, e_m) dt, \quad (2.21)$$

where

$$du = (-H \partial_x^2 u - u \partial_x u + \alpha \partial_x^2 u) dt + \sqrt{\alpha} d\zeta, \quad \zeta(t, x) = \sum_{m \in \mathbb{Z}_0} \lambda_m e_m(x) \beta_m(t).$$

Since  $E_n(u)$  is preserved by the BO equation, we have

$$\partial_u E_n(u, -H \partial_x^2 u - u \partial_x u) = 0.$$

By Lemma 2.2, we have

$$\partial_u E_n(u, \alpha \partial_x^2 u) \leq -\alpha \|u\|_{n+1}^2 + \alpha P_n(\|u\|), \quad (2.22)$$

$P_n$  is the polynomial of Lemma 2.2. Following the arguments of the proof of Lemma 2.2, we establish that

$$|\partial_u^2 E_n(u, e_m)| \leq c_n m^{2n} (\|u\|_n^2 + Q_n(\|u\|)), \quad (2.23)$$

where  $Q_n(r) = q_n r(1+r)^{k_n}$ ,  $q_n$  and  $k_n$  depend only on  $n$ . It remains to integrate (2.21) in  $t$ , take the expectation then the stochastic integral vanishes, and to combine (2.22) with (2.23) to get the claim.  $\square$

**Proposition 2.5.** *Let  $u$  be the solution of (2.19).*

1. *Suppose that  $\mathbb{E}E_0(u) < \infty$ , then*

$$\mathbb{E}E_0(u) + 2\alpha \int_0^t \mathbb{E}\|u(s)\|_1^2 ds = \mathbb{E}E_0(u_0) + \alpha A_0 t. \quad (2.24)$$

2. *Let  $p \geq 1$ . Suppose that  $\mathbb{E}E_0^p(u_0) < \infty$ , then*

$$\mathbb{E}E_0^p(u) \leq e^{-p\alpha t} \mathbb{E}E_0^p(u_0) + p^p A_0^p. \quad (2.25)$$

*Proof.* The identity (2.24) is easily proven applying the Ito formula to the conservation law  $E_0(u)$ . Let us prove (2.25) :

For  $p > 1$ , we apply the Ito formula to  $E_0^p(u)$  to find

$$dE_0^p(u) = pE_0^{p-1}(u)dE_0(u) + \frac{\alpha p(p-1)}{2} E_0^{p-2}(u) \sum_{m \in \mathbb{Z}_0} \lambda_m^2 |E'_0(u, e_m)|^2 dt.$$

Taking the expectation, we get

$$\mathbb{E}E_0^p(u) + \mathbb{E} \int_0^t f_\alpha(u(s)) ds = \mathbb{E}E_0^p(u_0),$$

where

$$f_\alpha(u) = \underbrace{2p\alpha E_0^{p-1}(u)\|u\|_1^2 - \alpha p E_0^{p-1}(u)A_0 - \frac{\alpha p(p-1)}{2} E_0^{p-2}(u) \sum_{m \in \mathbb{Z}_0} \lambda_m^2 |E'_0(u, e_m)|^2}_{\alpha Q}.$$

Now we see easily that

$$\sum_{m \in \mathbb{Z}_0} \lambda_m^2 |E'_0(u, e_m)|^2 \leq 2A_0 E_0(u).$$

Then applying Young inequality we find

$$Q \geq -\varepsilon E_0^p(u) - \frac{p^{2p}}{\varepsilon^{p-1}} A_0^p.$$

On the other hand

$$p\alpha E_0^{p-1}(u) \|u\|_1^2 \geq p\alpha E_0^p(u).$$

Choosing  $\varepsilon = p$ , we see that

$$f\alpha(u) \geq p\alpha \mathbb{E} E_0^p(u) - p^{p+1} A_0^p \alpha.$$

Then

$$\mathbb{E} E_0^p(u) + p\alpha \int_0^t \mathbb{E} E_0^p(u(s)) ds \leq \mathbb{E} E_0^p(u_0) + p^{p+1} A_0^p \alpha t.$$

Gronwall lemma gives the claimed result.  $\square$

### 3 Wellposedness of the stochastic BOB equation

**Proposition 3.1.** *Let  $s \geq 2$ . Suppose that  $A_s < \infty$ , then the problem (2.19) is globally wellposed in  $\dot{H}^s(\mathbb{T})$  in the sense that for all  $T > 0$*

1. *for any random variable  $u_0$  in  $\dot{H}^s(\mathbb{T})$  we have, for almost all  $\omega \in \Omega$ ,*

(a) *(Existence) there is  $u := u^\omega \in \Lambda_T(s) := C(0, T; \dot{H}^s) \cap L^2(0, T; \dot{H}^{s+1})$  satisfying the following relation in  $H^{s-2}$*

$$u(t) = u_0 - \int_0^t (H \partial_x^2 u + u \partial_x u - \alpha \partial_x^2 u) ds + \zeta(t) \text{ for all } t \in [0, T], \quad (3.1)$$

(b) *(Uniqueness) if  $u_1, u_2 \in \Lambda_T(s)$  are two solutions in the sense of (3.1) then  $u_1 \equiv u_2$  on  $[0, T]$ ,*

2. *the process solution  $u : (t, \omega) \mapsto u^\omega(t)$  is progressively measurable w.r.t.  $\sigma(u_0, \mathcal{F}_t)$ ,*

3. *(Continuity w.r.t. the initial data) for  $u_1$  and  $u_2$  two solutions of (1.6) starting at  $u_{1,0}$  and  $u_{2,0}$  respectively, we have almost surely*

$$\lim_{u_{1,0} \rightarrow u_{2,0}} u_1 = u_2 \text{ in } \Lambda_T(s). \quad (3.2)$$

In order to prove the existence result in Proposition 3.1, we split the problem (2.19) as follow:

- A linear stochastic problem:

$$\begin{cases} \partial_t z_\alpha + H \partial_x^2 z_\alpha = \alpha \partial_x^2 z_\alpha + \sqrt{\alpha} \eta & t > 0, \\ z_\alpha|_{t=0} = 0. \end{cases} \quad (3.3)$$

- A nonlinear deterministic problem:

$$\begin{cases} \partial_t v + H \partial_x^2 v + (v + z_\alpha) \partial_x (v + z_\alpha) = \alpha \partial_x^2 v & t > 0, \\ v|_{t=0} = u_0. \end{cases} \quad (3.4)$$

Here  $z_\alpha$  is a realization of the solution of (3.3).

For  $z_\alpha$  and  $v$  respective solutions of (3.3) and (3.4), it is easy to see that  $u = v + z_\alpha$  is solution of (2.19). The linear problem (3.3) is solved by the stochastic convolution (see [KS12, DPZ14])

$$z_\alpha(t) = \sqrt{\alpha} \int_0^t e^{-(t-s)(H-\alpha)\partial_x^2} d\zeta(s) =: \sqrt{\alpha} z(t). \quad (3.5)$$

If  $A_s$  is finite, we have

$$z \in C(\mathbb{R}_+, \dot{H}^s(\mathbb{T})) \cap L_{loc}^2(\mathbb{R}_+, \dot{H}^{s+1}(\mathbb{T})) \quad \text{for } \mathbb{P} - a.e. \quad \omega \in \Omega. \quad (3.6)$$

Uniqueness of solution for the problem (3.3) is obtained by standard arguments.

**Proposition 3.2.** *Let  $s \geq 2$ , suppose  $A_s < \infty$ . Let  $u_0$  be a random variable in  $\dot{H}^s(\mathbb{T})$ . Then for any  $T > 0$ , for a.e  $\omega$ , the nonlinear problem (3.4) associated to  $u_0$  admits a global solution in  $\Lambda_T(s)$ . Moreover this solution is adapted to  $\mathcal{F}_t$ .*

Proposition 3.2 is proved combining the two paragraphs below:

**A priori estimates.** The following lemma is proved using the first three integer order (modified) conservation laws  $E^*(u)$  of the remark 2.3, its proof is presented in the appendix.

**Lemma 3.3.** *For any  $T > 0$ , for almost any realization of  $z$  we have the following a priori estimates for the nonlinear problem (3.4)*

$$\sup_{t \in [0, T]} \|v(t)\|_i^2 + \alpha \int_0^T \|v(t)\|_{i+1}^2 dt \leq C \left( T, \|u_0\|_i, \|z\|_{L^\infty(0, T; H^i)} \right) \quad i = 0, 1, 2, \quad (3.7)$$

where  $C$  does not depend on  $\alpha \in (0, 1)$ .

Since  $H^2(\mathbb{T})$  is continuously embedded in  $C^1(\mathbb{T})$ , we infer

**Corollary 3.4.** *For any  $T > 0$ . For almost any realization of  $z$ , for any initial datum  $u_0 \in H^2$ , a solution  $v$  to (3.4) satisfies*

$$\sup_{t \in [0, T]} \|\partial_x v(t)\|_{L^\infty} \leq C \left( T, \|u_0\|_2, \|z\|_{L^\infty(0, T; H^2)} \right), \quad (3.8)$$

where  $C$  does not depend on  $\alpha \in (0, 1)$ .

**Lemma 3.5.** *For any  $T > 0$ , any  $s > 2$ , for almost any realization of  $z$  we have the high order a priori estimates for (3.4)*

$$\sup_{t \in [0, T]} \|v(t)\|_s^2 + \alpha \int_0^T \|v(t)\|_{s+1}^2 dt \leq C \left( T, \|u_0\|_s, \|z\|_{L^\infty(0, T; H^s)} \right), \quad (3.9)$$

where  $C$  does not depend on  $\alpha \in (0, 1)$ .

*Proof.* We recall the non-linear equation satisfied by  $v$ :

$$\partial_t v + H \partial_x^2 v - \alpha \partial_x^2 v = -v \partial_x v - \partial_x(v z_\alpha) - \frac{1}{2} \partial_x z_\alpha^2. \quad (3.10)$$

Then for  $s > 2$ , we have

$$(D^s v, D^s \partial_t v) + \alpha (D^{s+1} v, D^{s+1} v) = -(D^s v, D^s(v \partial_x v)) - (D^s v, D^s(v z_\alpha)) + \frac{1}{2} (D^{s+1} v, D^s z_\alpha^2).$$

Therefore

$$\frac{1}{2} \partial_t \|v\|_s^2 + \alpha \|v\|_{s+1}^2 = (1) + (2) + (3).$$

Using the Kato-Ponce commutator estimate (see the appendix of [IK09]) and the algebra structure of  $H^s(\mathbb{T})$ , we have

$$\begin{aligned} |(1)| &= |(D^s v, D^s(v \partial_x v)) - (v D^s \partial_x v) + (D^s v, v D^s \partial_x v)| \\ &= |(D^s v, [D^s, v] \partial_x v) - \frac{1}{2} (\partial_x v, |D^s v|^2)| \\ &\leq C \|v\|_s^2 \|\partial_x v\|_{L^\infty}. \end{aligned}$$

It is not difficult to establish

$$|(2)| + |(3)| \leq \frac{\alpha}{2} \|v\|_{s+1}^2 + C_1 \|v\|_s^2 \|z\|_s^2 + C_2 \alpha \|z\|_s^4, \quad (3.11)$$

where  $C_1$  and  $C_2$  are universal. It remains to combine the Gronwall lemma with (3.8) to get the claim.  $\square$

**Local and global existence for the nonlinear problem (3.4).** Let  $s \geq 2$ . For a positive  $T$  the space  $\Lambda_T(s)$  is endowed with the norm defined by

$$\|u\|_{\Lambda_T(s)} = \sup_{t \in [0, T]} \left( e^{-\frac{t}{T}} \left\{ \|u(t)\|_s^2 + \alpha \int_0^t \|u(r)\|_{s+1}^2 dr \right\} \right)^{\frac{1}{2}}. \quad (3.12)$$

Let  $R > 0$ , denote by  $B_R$  the ball in  $H^s$  of center 0 and radius  $R$ .

**Proposition 3.6.** *Let  $s \geq 2$ ,  $\alpha \in (0, 1)$ . For all  $R > 0$ , there is  $T_R > 0$  such that for any  $u_0$  in  $B_{R/2}$ , the nonlinear problem (3.4) has a unique solution in  $\Lambda_{T_R}(s)$ .*

*Remark 3.7.* Since the time existence of Proposition 3.6 depends only on the size of the initial data, we can do an iteration to get the global existence for (3.4).

*Proof of Proposition 3.6.* Let us look for a fixed point of the following map

$$\mathfrak{F}w = e^{-t(H-\alpha)\partial_x^2} u_0 - \int_0^t e^{-(t-s)(H-\alpha)\partial_x^2} (z_\alpha + w) \partial_x(z_\alpha + w) ds.$$

We proceed as follow:

- **Step 1:** We prove that for any  $R > 0$ , there is  $T > 0$  such that the ball of  $\Lambda_T(s)$  centered at 0 and of radius  $R$  is invariant under  $\mathfrak{F}$  if  $\|u_0\|_s \leq R/2$ .

$$\begin{aligned}
-\frac{1}{2} \frac{d}{dt} \|\mathfrak{F}w\|_s^2 &= -(\partial_t D^s \mathfrak{F}w, D^s \mathfrak{F}(w)) \\
&= -((H - \alpha) D^{s+1} \mathfrak{F}(w), D^{s+1} \mathfrak{F}(w)) + \frac{1}{2} (D^s (z_\alpha + w)^2, D^{s+1} \mathfrak{F}(w)) \\
&\geq \alpha \|\mathfrak{F}(w)\|_{s+1}^2 - \frac{1}{2} \|z_\alpha + w\|_s^2 \|\mathfrak{F}(w)\|_{s+1} \\
&\geq \alpha \|\mathfrak{F}(w)\|_{s+1}^2 - \frac{\alpha}{2} \|\mathfrak{F}(w)\|_{s+1}^2 - \frac{1}{8\alpha} (\|z_\alpha\|_s^4 + \|w\|_s^4).
\end{aligned}$$

Then there is an universal constant  $c > 0$  such that

$$\frac{d}{dt} \|\mathfrak{F}(w)\|_s^2 + \alpha \|\mathfrak{F}(w)\|_{s+1}^2 \leq \frac{c}{\alpha} e^{\frac{2}{T}} (R^4 + \|z_\alpha\|_{\Lambda_T(s)}^4).$$

Thus, after integration with respect to  $t$ , we find

$$\|\mathfrak{F}(w)\|_s^2 + \alpha \int_0^t \|\mathfrak{F}(w)\|_{s+1}^2 ds \leq \|u_0\|_s^2 + \frac{\tilde{c}T}{\alpha} e^{\frac{1}{T}} (R^4 + \|z_\alpha\|_{\Lambda_T(s)}^4).$$

Multiplying the last relation by  $e^{-\frac{1}{T}}$ , it remains to choose  $T$  so much so that we obtain the claimed result.

- Step 2: We now prove that  $\mathfrak{F}$  is a contraction on the ball constructed above. We have

$$\partial_t \mathfrak{F}w = -\{(w + z_\alpha) \partial_x (w + z_\alpha) + (H - \alpha) \partial_x^2 \mathfrak{F}w\},$$

then for  $w_1$  and  $w_2$  in  $\Lambda_T(s)$ , we have

$$\begin{aligned}
-\frac{1}{2} \frac{d}{dt} \|\mathfrak{F}w_1 - \mathfrak{F}w_2\|_s^2 &= (\partial_t D^s (\mathfrak{F}w_1 - \mathfrak{F}w_2), D^s (\mathfrak{F}w_1 - \mathfrak{F}w_2)) \\
&= -(D^s (F_z(w_1) - F_z(w_2)), D^{s+1} (\mathfrak{F}w_1 - \mathfrak{F}w_2)) \\
&\quad + \alpha \|\mathfrak{F}w_1 - \mathfrak{F}w_2\|_{s+1}^2,
\end{aligned}$$

where

$$F_z(w) = \frac{1}{2} (z_\alpha + w)^2.$$

We show easily that

$$\|D^s (F_z(w_1) - F_z(w_2))\|^2 \leq C(s) \|w_1 - w_2\|_s^2 (\|w_1 + w_2\|_s^2 + \|z_\alpha\|_s^2),$$

this allows to get that

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\mathfrak{F}w_1 - \mathfrak{F}w_2\|_s^2 + \frac{\alpha}{2} \|\mathfrak{F}w_1 - \mathfrak{F}w_2\|_{s+1}^2 &\leq \frac{C(s)}{\alpha} \|w_1 - w_2\|_s^2 (\|w_1 + w_2\|_s^2 + \|z_\alpha\|_s^2) \\
&\leq e^{\frac{1}{T}} \frac{C(s) (4R^2 + \|z\|_{\Lambda_T(s)}^2)}{\alpha} \|w_1 - w_2\|_{\Lambda_T(s)}^2.
\end{aligned}$$

After integration in  $t$ , we find

$$\|\mathfrak{F}w_1 - \mathfrak{F}w_2\|_s^2 + \alpha \int_0^t \|\mathfrak{F}w_1 - \mathfrak{F}w_2\|_{s+1}^2 ds \leq T e^{\frac{1}{T}} \frac{C(s) (4R^2 + \|z_\alpha\|_{\Lambda_T(s)}^2)}{\alpha} \|w_1 - w_2\|_{\Lambda_T(s)}^2.$$

We multiply this inequality by  $e^{-\frac{1}{T}}$ , the  $T$  found in the first step can be decreased if necessary to give an contraction.

The fixed point theorem allows to conclude.  $\square$

*Remark 3.8.* By definition,  $v$  is  $\sigma(u_0, \mathcal{F}_t)$ -adapted.

**End of the proof of wellposedness of (1.6).**

*End of the proof of Proposition 3.1.* The function  $u = v + z_\alpha$  is a  $\sigma(u_0, \mathcal{F}_t)$ -adapted solution of (1.6), then, at least, the process  $u$  has a progressive modification (see [KS91]). Let  $u_1$  and  $u_2$  be two solutions of (1.6) starting respectively at  $u_{1,0}$  and  $u_{2,0}$ , set  $w = u_1 - u_2$ , then the problem solved by  $w$  is

$$\begin{cases} \partial_t w + (H - \alpha) \partial_x^2 w + w \partial_x w + \partial_x(w u_2) = 0, \\ w|_{t=0} = u_{1,0} - u_{2,0} =: w_0. \end{cases} \quad (3.13)$$

Using the arguments of the proof of (3.9), we show

$$\sup_{t \in [0, T]} \|w(t)\|_s^2 + \alpha \int_0^T \|w(r)\|_{s+1}^2 dr \leq C(\alpha, T, \|\partial_x w\|_{L^\infty(0, T; L^\infty)}, \|u_2\|_{L^\infty(0, T; H^s)}) \|w_0\|_s^2. \quad (3.14)$$

This completes the proof.  $\square$

**Some probabilistic estimates for the linear problem.** Suppose  $A_n$  finite. Application of the Ito formula to the  $H^n$ -norms (which are preserved by the linear Benjamin-Ono) shows that

$$\mathbb{E} \|z_\alpha\|_n^2 + 2\alpha \int_0^t \mathbb{E} \|z_\alpha\|_{n+1}^2 ds = \alpha A_n t. \quad (3.15)$$

## 4 Invariant measures for the viscous problem

Consider the stochastic BOB problem (1.6) posed on  $\dot{H}^2(\mathbb{T})$ . By the estimates (2.24), (2.25) and (2.4), we have

$$\begin{aligned} \mathbb{E} E_0(u) + 2\alpha \int_0^t \mathbb{E} \|u\|_1^2 ds &= \mathbb{E} E_0(u_0) + \alpha A_0 t, \\ \mathbb{E} E_0^p(u) &\leq e^{-p\alpha t} \mathbb{E} E_0^p(u_0) + C_p A_0^p, \\ \mathbb{E} E_1(u) + \alpha \int_0^t \mathbb{E} \|u\|_2^2 ds &\leq \mathbb{E} E_1(u_0) + \alpha \left[ A_1 t + c_1 \int_0^t \mathbb{E} \|u\|_1^2 ds + \int_0^t \mathbb{E} W_1(\|u\|) ds \right], \\ \mathbb{E} E_2(u) + \alpha \int_0^t \mathbb{E} \|u\|_3^2 ds &\leq \mathbb{E} E_2(u_0) + \alpha \left[ A_2 t + c_2 \int_0^t \mathbb{E} \|u\|_2^2 ds + \int_0^t \mathbb{E} W_2(\|u\|) ds \right]. \end{aligned}$$

Recall that  $W_1$  and  $W_2$  are the polynomials of (2.4), their expectation are controlled using the second estimate. Now suppose  $u_0 = 0$  almost surely, then by a recurrence process, we get

$$\mathbb{E} E_2(u) + \alpha \int_0^t \mathbb{E} \|u\|_3^2 ds \leq \alpha C t, \quad (4.1)$$

where  $C$  is universal. Now in view of Remark 2.3, we can suppose  $E_n(u) \geq 0$  (indeed, adding  $c\|u\|^6$  to  $E_2(u)$  we find similar estimate). Then

$$\frac{1}{t} \int_0^t \mathbb{E} \|u\|_3^2 ds \leq C, \quad (4.2)$$

where  $C$  is, in particular, independent of  $t$ . Denote by  $\lambda_\alpha(t)$  the law of the solution  $u(t)$  to (1.6) starting at 0, and consider the time average

$$\bar{\lambda}_\alpha(t) = \frac{1}{t} \int_0^t \lambda_\alpha(s) ds. \quad (4.3)$$

Using the estimate (4.2), we show

$$\int_{H^2} \|u\|_3^2 \bar{\lambda}_\alpha(t)(du) \leq C. \quad (4.4)$$

The compactness of the embedding  $H^3(\mathbb{T}) \subset H^2(\mathbb{T})$ , the Chebyshev inequality, the Prokhorov theorem and the classical Bogoliubov-Krylov argument imply the following result:

**Proposition 4.1.** *For any  $\alpha \in (0,1)$ , the stochastic BOB equation (1.6) posed in  $H^2(\mathbb{T})$  admits an invariant measure  $\mu_\alpha$  concentrated on  $H^3(\mathbb{T})$ .*

**Theorem 4.2.** *Let  $\alpha \in (0,1)$ . Suppose that  $A_n$  is finite for any  $n$ . Then any invariant measure  $\mu_\alpha$  of the problem (1.6) posed in  $\dot{H}^2(\mathbb{T})$  satisfies*

$$\int_{H^2(\mathbb{T})} \|u\|_1^2 \mu_\alpha(du) = \frac{A_0}{2}, \quad (4.5)$$

$$\int_{H^2(\mathbb{T})} \|u\|^{2p} \mu_\alpha(du) \leq p^p A_0^p \text{ for any } 1 \leq p < \infty, \quad (4.6)$$

$$\int_{H^2(\mathbb{T})} \|u\|_n^2 \mu_\alpha(du) \leq D_n \text{ for any } n \geq 2, \quad (4.7)$$

where  $D_n$  depends only on  $n$ .

*Proof.* It suffices to prove the estimate (4.6). Indeed, we then have in particular,

$$\mathbb{E} E_0(u) \leq C, \quad (4.8)$$

so we can deduce (4.5) from (2.24) using the invariance of  $\mu_\alpha$ . Now since

$$\mathbb{E} E_n(u) \leq \mathbb{E} \|u\|_n^2 + c_n^+ \mathbb{E} \|u\|^{2n+2}, \quad (4.9)$$

once  $\mathbb{E} \|u\|_n^2$  estimated, we are able to use (2.20) and to gain the estimation of  $\mathbb{E} \|u\|_{n+1}^2$ . This recurrence process gives (4.7). Let us prove (4.6):

Let  $R > 0$ , consider a  $C^\infty$ -function  $\chi_R$  satisfying

$$\chi_R(u) = \begin{cases} 1, & \text{if } \|u\|_2 \leq R, \\ 0, & \text{if } \|u\|_2 > R+1. \end{cases} \quad (4.10)$$

Let  $p \geq 1$ , we have

$$\int_{H^2} E_0^p(u) \chi_R(u) \mu_\alpha(du) = \int_{H^2} \mathbb{E} \{E_0^p(u(t,v)) \chi_R(u(t,v))\} \mu_\alpha(dv), \quad (4.11)$$

where  $u(\cdot, v)$  is the solution of (1.6) starting at  $v$ . We pass to the limit  $t \rightarrow \infty$  in the right hand side of (4.11) using (2.25) ( $u$  is in the ball of size  $R$ ) and the invariance of  $\mu_\alpha$ , we find

$$\int_{H^2} E_0^p(u) \chi_R(u) \mu_\alpha(du) \leq p^p A_0^p. \quad (4.12)$$

Now Fatou's Lemma allows to conclude.  $\square$

**Corollary 4.3.** *Let  $\alpha \in (0, 1)$ . Suppose  $A_n < \infty$  for any  $n$ . Then any invariant measure  $\mu_\alpha$  for the stochastic BOB problem (1.6) posed in  $\dot{H}^2(\mathbb{T})$  is concentrated on  $C^\infty(\mathbb{T})$ .*

*Proof.* Let  $n > 2$ . Combining the estimate (4.7) and the Chebyshev inequality we find

$$\mu_\alpha(\{u \in H^2 : \|u\|_n^2 \geq R\}) \leq \frac{D_n}{R^2}.$$

Set  $B_n(0, R)$  the ball in  $H^n$  of center 0 and radius  $R$ , we have

$$\int_{H^2} \mathbb{1}_{B_n(0, R)}(u) \mu_\alpha(du) = \mu_\alpha(B_n(0, R)) \geq 1 - \frac{D_n}{R^2}.$$

Passing to the limit on  $R$  (with use Lebesgue convergence theorem), we get

$$\mu_\alpha(H^n(\mathbb{T})) = 1. \quad (4.13)$$

Thus

$$1 = \mu_\alpha(\cap_{n>2} H^n(\mathbb{T})) = \mu_\alpha(C^\infty(\mathbb{T})). \quad (4.14)$$

□

## 5 Invariant measure for the BO equation

In this section,  $S_t : H^3(\mathbb{T}) \rightarrow H^3(\mathbb{T})$ ,  $t \geq 0$ , denotes the flow of the Benjamin-Ono equation (1.1). The map  $S_{t,\alpha} : H^3 \rightarrow H^3$  denotes the one of the stochastic Benjamin-Ono-Burgers equation (1.6). We define the associated Markov semi-group:

$$\begin{aligned} \phi_t f(w) &= \mathbb{E}[f(S_t w)], \\ \phi_{t,\alpha} f(w) &= \mathbb{E}[f(S_{t,\alpha} w)], \end{aligned}$$

where  $f$  is a real bounded Lipschitz function on  $H^3(\mathbb{T})$  and  $w \in H^3(\mathbb{T})$ . We denote by  $\phi_t^*$  and  $\phi_{t,\alpha}^*$  the dual maps of  $\phi_t$  and  $\phi_{t,\alpha}$ :

$$\begin{aligned} \phi_t^* v(f) &= \int_{H^3} f(w) v(dw), \\ \phi_{t,\alpha}^* v(f) &= \int_{H^3} f(w) v(dw), \end{aligned}$$

for  $v$  a probability measure on  $H^3$  and  $f$  a real bounded continuous function on  $H^3$ .

### 5.1 Some convergence results of stochastic BOB to BO

**Lemma 5.1.** *For any  $T > 0$ . For  $\mathbb{P}$ -almost any  $w \in H^3(\mathbb{T})$ , we have*

$$\sup_{t \in [0, T]} \|S_{t,\alpha} w - S_t w\|_2 \rightarrow 0 \quad \text{as } \alpha \rightarrow 0. \quad (5.1)$$

*Proof.* We write

$$\|S_{t,\alpha} w - S_t w\|_2 = \|v + z - S_t w\|_2 \leq \|v - S_t w\|_2 + \|z\|_2,$$

where

$$z_\alpha(t) = \sqrt{\alpha} \int_0^t e^{-(t-s)(H-\alpha)\partial_x^2} d\zeta(s) = \sqrt{\alpha} z(t)$$



and  $v$  is the solution of

$$\partial_t v + H \partial_x^2 v + (v + z_\alpha) \partial_x (v + z_\alpha) = \alpha \partial_x^2 v \quad (5.2)$$

$$v_{t=0} = w. \quad (5.3)$$

Then almost surely  $\sup_{t \in [0, T]} \|z_\alpha\|_2 = O(\sqrt{\alpha})$  using, in particular, the continuity of  $z$  in  $t$ . Set  $h = v - S_t w$ , we have

$$\sup_{t \in [0, T]} \|S_{t, \alpha} w - S_t w\|_2 \leq \sup_{t \in [0, T]} \|h\|_2 + O(\sqrt{\alpha}). \quad (5.4)$$

We claim that  $\sup_{t \in [0, T]} \|h\|_2 = O(\sqrt{\alpha})$ . Indeed using the estimate (3.9) and the  $H^3$ -conservation law, we show that

$$\|h\|_2^3 \leq c \|h\| \|h\|_3^2 \leq C(T, \|w\|_{L^\infty(0, T; H^3)}, \|z\|_{L^\infty(0, T; H^3)}) \|h\|. \quad (5.5)$$

Taking the difference between (5.2) and the BO equation (1.1), we see that  $h$  satisfies

$$\partial_t h + H \partial_x^2 h + h \partial_x h = -\partial_x (h S_t w) + O(\sqrt{\alpha}). \quad (5.6)$$

Thanks to the conservation of the  $L^2$ -norm by BO and the embedding  $H^{\frac{3}{2}+}(\mathbb{T}) \subset C^1(\mathbb{T})$ , we get

$$\partial_t \|h\|^2 \leq \|h\|^2 \|S_t w\|_{L^\infty(0, T; H^{3/2+})} + C(T, \|h\|_{L^\infty(0, T; H^2)}, \|z\|_{L^\infty(0, T; H^2)}) \sqrt{\alpha}.$$

Using the  $H^2$ -conservation law, we control  $\|S_t w\|_{L^\infty(0, T; H^{3/2+})}$ . It remains to apply the Gronwall lemma to get the claim.  $\square$

**Lemma 5.2.** *For all  $T, R, r > 0$ , we have*

$$\sup_{w \in B(0, R)} \sup_{t \in [0, T]} \mathbb{E} \left[ \|S_{t, \alpha} w - S_t w\|_2 \mathbb{1}_{\{\|z\|_{L^\infty(0, T; H^2)} \leq r\}} \right] = O_{R, r, T}(\sqrt{\alpha}). \quad (5.7)$$

Here  $B(0, R)$  is the ball in  $H^3(\mathbb{T})$  of center 0 and radius  $R$ .

*Proof.*

$$\begin{aligned} \mathbb{E} \left[ \|S_{t, \alpha} w - S_t w\|_2 \mathbb{1}_{\{\|z\|_{L^\infty(0, T; H^2)} \leq r\}} \right] &= \int_{\Omega} \|S_{t, \alpha} w - S_t w\|_2 \mathbb{1}_{\{\|z\|_{L^\infty(0, T; H^2)} \leq r\}}(\omega) d\mathbb{P}(\omega) \\ &\leq \int_{\Omega} [\|h\|_2 + r \sqrt{\alpha}] \mathbb{1}_{\{\|z\|_{L^\infty(0, T; H^2)} \leq r\}}(\omega) d\mathbb{P}(\omega), \end{aligned}$$

where  $h = v - S_t w$  as previously. The arguments of the proof of Lemma 5.1 allow to see that  $\sup_{t \in [0, T]} \|h\|_2 \leq C_{R, r, T} \sqrt{\alpha}$ . This gives the claimed result.  $\square$

## 5.2 An accumulation point for the viscous measures

In what follow we denote by  $M(H^3)$  the space of probability measures on  $H^3$ .

**Theorem 5.3.** *There is an accumulation point  $\mu$  of the sequence  $\{\mu_\alpha, \alpha \in (0, 1)\}$  in  $M(H^3)$  satisfying:*

- $\mu$  is invariant under the flow of the Benjamin-Ono equation in  $H^3(\mathbb{T})$ ,
- $\mu$  is concentrated on  $C^\infty(\mathbb{T})$ .

- The measure  $\mu$  satisfies

$$\int_{H^3(\mathbb{T})} \|u\|_1^2 \mu(du) = \frac{A_0}{2}, \quad (5.8)$$

$$\int_{H^3(\mathbb{T})} \|u\|^{2p} \mu(du) \leq p^p A_0^p \text{ for any } 1 \leq p < \infty, \quad (5.9)$$

$$\int_{H^3(\mathbb{T})} \|u\|_n^2 \mu(du) < \infty \text{ for } n \geq 2. \quad (5.10)$$

*Proof.* The proof consists in the following four steps:

1. **Existence of an accumulation point  $\mu$ .** The estimate (4.7) with  $n = 4$  implies the tightness of the sequence of measures  $(\mu_\alpha)$  in  $H^3(\mathbb{T})$  and so the existence of an accumulation point  $\mu$  on  $H^3(\mathbb{T})$ .

2. **Invariance of  $\mu$  under the Benjamin-Ono flow.** Denote by  $(\mu_{\alpha_k})_{k \in \mathbb{N}}$  a subsequence of  $(\mu_\alpha)$  converging to  $\mu$ , to simplify the notations we write  $\mu_k$  instead. The corresponding flow and Markov semi-group will be denoted  $S_{t,k}$  and  $\phi_{t,k}$ . The following diagram represents the idea of proof of the invariance of  $\mu$ :

$$\begin{array}{ccc} \phi_{t,k}^* \mu_k & \xlongequal{(I)} & \mu_k \\ \downarrow (III) & & \downarrow (II) \\ \phi_t^* \mu & \xlongequal{(IV)} & \mu \end{array}$$

The equality (I) is the invariance of  $\mu_k$  by  $\phi_{t,k}$ , (II) is proved above. Then (IV) is proved once (III) is checked.

Let  $f$  be a real bounded Lipschitz function on  $H^2(\mathbb{T})$ . Without loss of generality assume that  $f$  is bounded by 1. Then

$$\begin{aligned} (\phi_{t,k}^* \mu_k, f) - (\phi_t^* \mu, f) &= (\mu_k, \phi_{t,k} f) - (\mu, \phi_t f) \\ &= \underbrace{(\mu_k, \phi_{t,k} f - \phi_t f)}_A - \underbrace{(\mu - \mu_k, \phi_t f)}_B. \end{aligned}$$

The term  $B$  converges to 0 as  $k \rightarrow \infty$  by the weak convergence of  $(\mu_k)$  to  $\mu$ . And for any  $R > 0$

$$\begin{aligned} |A| &\leq \int_{H^3} \mathbb{E} |f(S_{t,k} w) - f(S_t w)| \mu_k(dw) \\ &= \underbrace{\int_{B(0,R)} \mathbb{E} |f(S_{t,k} w) - f(S_t w)| \mu_k(dw)}_{A_1} + \underbrace{\int_{H^3 \setminus B(0,R)} \mathbb{E} |f(S_{t,k} w) - f(S_t w)| \mu_k(dw)}_{A_2}. \end{aligned}$$

Recalling that  $f$  is bounded by 1, we get by the Chebyshev inequality

$$A_2 \leq 2\mu_k(H^3 \setminus B(0,R)) \leq \frac{C}{R^2}, \quad (5.11)$$

where  $C$  is finite and does not depend on  $k$  (estimate (4.7)). Denote by  $L_t^\infty H_x^2$  the space  $L^\infty(0, T; H^2)$ . Let  $r > 0$ , we have

$$\begin{aligned} A_1 &= \underbrace{\int_{B(0,R)} \mathbb{E} \left[ |f(S_{t,k}w) - f(S_t w)| \mathbb{1}_{\{\|z\|_{L_t^\infty H_x^2} \leq r\}} \right] \mu_k(dw)}_{A_{1,1}} \\ &\quad + \underbrace{\int_{B(0,R)} \mathbb{E} \left[ |f(S_{t,k}w) - f(S_t w)| \mathbb{1}_{\{\|z\|_{L_t^\infty H_x^2} > r\}} \right] \mu_k(dw)}_{A_{1,2}}. \end{aligned}$$

As previously since  $f$  is bounded by 1, with use of (3.15) (with  $\alpha = 1$ ) the Chebyshev inequality implies

$$A_{1,2} \leq \frac{C_T}{r^2}. \quad (5.12)$$

On the other hand  $f$  being Lipschitz on  $H^2$ , denote by  $C_f$  its Lipschitz constant, we have

$$\begin{aligned} A_{1,1} &\leq C_f \int_{B(0,R)} \mathbb{E} \left[ \|S_{t,k}w - S_t w\|_2 \mathbb{1}_{\{\|z\|_{L_t^\infty H_x^2} \leq r\}} \right] \mu_k(dw) \\ &\leq C_f \sup_{w \in B(0,R)} \mathbb{E} \left[ \|S_{t,k}w - S_t w\|_2 \mathbb{1}_{\{\|z\|_{L_t^\infty H_x^2} \leq r\}} \right]. \end{aligned}$$

According to Lemma 5.2, we find

$$A_{1,1} \leq C_{f,R,r,T} \sqrt{\alpha_k}. \quad (5.13)$$

Finally, we arrive at

$$|A| \leq C_{f,R,r,T} \sqrt{\alpha_k} + \text{Const}(T) \left( \frac{1}{r^2} + \frac{1}{R^2} \right), \quad (5.14)$$

where  $\text{Const}$  does not depend on  $k$ . We get the desired result after passing to the limits in this order

$$\begin{aligned} k &\rightarrow \infty, \\ R, r &\rightarrow \infty. \end{aligned}$$

**3. The estimates for the measure  $\mu$ .** Denote by  $\chi_R$  a bump function on the ball  $B(0, R)$  of  $H^3(\mathbb{T})$ , by (4.5) we have

$$\int_{H^3} \chi_R(v) \|v\|_1^2 \mu_k(dv) \leq \frac{A_0}{2}$$

Passing to the limit  $k \rightarrow \infty$  we find

$$\int_{H^3} \chi_R(v) \|v\|_1^2 \mu(dv) \leq \frac{A_0}{2}.$$

Then Fatou's lemma gives

$$\mathbb{E} \|u\|_1^2 = \int_{H^3} \|v\|_1^2 \mu(dv) \leq \frac{A_0}{2}. \quad (5.15)$$

We can do the same process to show (5.9) and (5.10). Now we write

$$\frac{A_0}{2} = \int_{B(0,R)} \|v\|_1^2 \mu_k(dv) + \int_{H^3 \setminus B(0,R)} \|v\|_1^2 \mu_k(dv).$$

We use the Cauchy-Schwarz and Chebyshev inequalities to show that

$$\begin{aligned} \int_{H^3 \setminus B(0,R)} \|u\|_1^2 \mu_k(du) &= \int_{H^3} \|u\|_1^2 \mathbb{1}_{\|u\|_3 > R}(u) \mu_k(du) \\ &\leq \left( \int_{H^3} \|u\|_1^4 \mu_k(du) \right)^{\frac{1}{2}} (\mu_k(\|u\|_3 > R))^{\frac{1}{2}} \\ &\leq \frac{\sqrt{\mathbb{E}[\|u\|_3^2] \mathbb{E}[\|u\|_1^4]}}{R}. \end{aligned}$$

We can control  $\mathbb{E}[\|u\|_1^4]$  and  $\mathbb{E}[\|u\|_3^2]$  uniformly in  $k$  combining interpolation inequalities and the estimates (5.9) and (5.10). Then there is a constant  $C > 0$  independent of  $k$  such that

$$\frac{A_0}{2} - \frac{C}{R} \leq \int_{H^3} \chi_R(v) \|v\|_1^2 \mu_k(dv).$$

We find (5.8) after passing to the limits in the order

$$\begin{aligned} k &\rightarrow \infty, \\ R &\rightarrow \infty, \end{aligned}$$

and combining this with (5.15).

**4. The measure  $\mu$  is concentrated on  $C^\infty(\mathbb{T})$ .** This immediately follows the estimates (5.10) using the arguments of the proof of Corollary 4.3.  $\square$

## 6 Qualitative properties of the measure

### 6.1 Absolute continuity of some observables w.r.t. to the Lebesgue measure

The following result is inspired by [Shi11, KS12] where the local time concept is used to deduce non-degeneracy properties of measures constructed for Schrödinger and Euler equations.

**Theorem 6.1.** *Suppose that  $\lambda_m \neq 0$  for all  $m$ . Then for any integer  $n \geq 1$ ,  $b_n$  and  $c_n$  such that the distribution of the observable  $\tilde{E}_n(u) := E_n(u) + c_n \|u\|^2 (1 + \|u\|^2)^{b_n}$  under  $\mu$  has a density w.r.t. the Lebesgue measure on  $\mathbb{R}$ .*

For the proof of Theorem 6.2 below, we refer the reader to [Shi11] and the proof of Theorem 5.2.12 of [KS12] where the authors prove similar results in the case of Schrödinger and Euler equations respectively. The proof is exactly the same.

**Theorem 6.2.** *The measure  $\mu$  constructed in Theorem 5.3 satisfies the following non-degeneracy properties:*

1. Let  $\lambda_m \neq 0$  for at least two indices. Then  $\mu$  has no atom at zero and

$$\mu(\{u \in C^\infty : \|u\| \leq \delta\}) \leq C\sqrt{A_0}\gamma^{-1}\delta \quad \text{for all } \delta > 0, \quad (6.1)$$

where  $\gamma = \inf\{A_0 - \lambda_m^2, m \in \mathbb{Z}\}$  and  $C$  is an universal constant.

2. Let  $\lambda_m \neq 0$  for all indices. Then there is an increasing continuous function  $h(r)$  vanishing at  $r = 0$  such that

$$\mu(\{u \in C^\infty(\mathbb{T}) : \|u\| \in \Gamma\}) \leq h(\lambda(\Gamma)) \quad (6.2)$$

for any Borel set  $\Gamma \subset \mathbb{R}$ , where  $\lambda$  stands for the Lebesgue measure on  $\mathbb{R}$ .

*Proof of Theorem 6.1.* We prove the claim for the viscous measures with bounds uniform in  $\alpha$ , then we can pass to the limit in the viscosity to obtain the wanted result (using the Portmanteau theorem). First we apply Ito formula to  $\tilde{E}_n(u)$ :

$$\tilde{E}_n(u(t)) = \tilde{E}_n(u(0)) + \alpha \int_0^t A(s)ds + \sqrt{\alpha} \sum_{m \in \mathbb{Z}_0} \lambda_m \int_0^t \tilde{E}_n'(u, e_m) d\beta_m(s), \quad (6.3)$$

where

$$A(s) = \partial_u \tilde{E}_n(u, \partial_x^2 u) + \frac{1}{2} \sum_{m \in \mathbb{Z}_0} \lambda_m^2 \partial_u^2 \tilde{E}_n(u, e_m). \quad (6.4)$$

Then we denote by  $\Lambda_t(a, \omega)$  its local time which reads (see Appendix A.8 of [KS12])

$$\begin{aligned} \Lambda_t(a, \omega) &= (\tilde{E}_n(u(t)) - a)_+ - (\tilde{E}_n(u(0)) - a)_+ - \alpha \int_0^t A(s) \mathbb{1}_{(a, +\infty)}(\tilde{E}_n(u)) ds \\ &\quad - \sqrt{\alpha} \sum_{m \in \mathbb{Z}_0} \lambda_m \int_0^t \mathbb{1}_{(a, +\infty)}(\tilde{E}_n(u)) \tilde{E}_n'(u, e_m) d\beta_m(s). \end{aligned}$$

Using the stationarity of  $u$ , we infer that

$$\mathbb{E} \Lambda_t(a) = -\alpha t \mathbb{E}[A(0) \mathbb{1}_{(a, +\infty)}(\tilde{E}_n(u))]. \quad (6.5)$$

Now using the well known identity of local time with the function  $\mathbb{1}_\Gamma$ , we get

$$2 \int_\Gamma \Lambda_t(a) da = \alpha \sum_{m \in \mathbb{Z}_0} \lambda_m^2 \int_0^t \mathbb{1}_\Gamma(\tilde{E}_n(u)) \tilde{E}_n'(u, e_m)^2 ds. \quad (6.6)$$

Hence by stationarity of  $u$ , we have

$$2 \int_\Gamma \mathbb{E} \Lambda_t(a) da = \alpha t \sum_{m \in \mathbb{Z}_0} \lambda_m^2 \mathbb{E}[\mathbb{1}_\Gamma(\tilde{E}_n(u)) \tilde{E}_n'(u, e_m)^2]. \quad (6.7)$$

Comparing (6.5) and (6.7), we find

$$\sum_{m \in \mathbb{Z}_0} \lambda_m^2 \mathbb{E}[\mathbb{1}_\Gamma(\tilde{E}_n(u)) \tilde{E}_n'(u, e_m)^2] \leq 2\lambda(\Gamma) \mathbb{E}|A(0)| \leq C\lambda(\Gamma). \quad (6.8)$$

Now we recall the general form of  $\tilde{E}_n(u)$

$$\tilde{E}_n(u) = \|u\|_n^2 + R_n(u) + P_n(\|u\|^2), \quad (6.9)$$

where

$$P_n(r) = c_n r(1+r)^{b_n}. \quad (6.10)$$

Then

$$\tilde{E}'_n(u, v) = 2(D^n u, D^n v) + R'_n(u, v) + 2(u, v)P'_n(\|u\|^2). \quad (6.11)$$

Recalling Remark 2.3, we have

$$\tilde{E}'_n(u, u) \geq \|u\|_n^2. \quad (6.12)$$

Now we define the operator  $A_n$  so that

$$\tilde{E}'_n(u, v) = (A_n u, v). \quad (6.13)$$

Therefore

$$\begin{aligned} (A_n u, u) &= \sum_{m \in \mathbb{Z}_0} u_m (A_n u, e_m) \\ &= \sum_{|m| \leq N} u_m (A_n u, e_m) + \sum_{|m| > N} u_m (A_n u, e_m) \\ &\leq \frac{\|u\|}{\underline{\lambda}_N} \left( \sum_{|m| \leq N} \lambda_m^2 (A_n u, e_m)^2 \right)^{\frac{1}{2}} + \|A_n u\| \left( \sum_{|m| > N} u_m^2 \right)^{\frac{1}{2}} \\ &\leq \frac{\|u\|_1}{\underline{\lambda}_N} \left( \sum_{m \in \mathbb{Z}_0} \lambda_m^2 \tilde{E}'_n(u, e_m)^2 \right)^{\frac{1}{2}} + \|A_n u\| \frac{\|u\|_1}{N}, \end{aligned}$$

where  $\underline{\lambda}_N = \min\{\lambda_m, |m| \leq N\}$ . We take into account (6.12) and consider  $u$  belonging to

$$A_\varepsilon = \left\{ v : \|v\| \geq \varepsilon, \|A_n v\| \leq \frac{1}{\varepsilon} \right\}. \quad (6.14)$$

We get

$$\sum_{m \in \mathbb{Z}_0} \lambda_m^2 \tilde{E}'_n(u, e_m)^2 \geq \underline{\lambda}_N^2 \left( \varepsilon - \frac{1}{N^2 \varepsilon} \right)^2. \quad (6.15)$$

The integer  $N$  can be chosen to depend on  $\varepsilon$  so that the function

$$\alpha(\varepsilon) := \left( \varepsilon - \frac{1}{N^2 \varepsilon} \right)^2$$

is positive, increasing and converges to 0 when  $\varepsilon \rightarrow 0$ . Then we have

$$\mu(\{u : \tilde{E}'_n(u) \in \Gamma\} \cap A_\varepsilon) \leq \frac{C}{\alpha(\varepsilon)} \lambda(\Gamma). \quad (6.16)$$

Consider now the complementary set

$$A_\varepsilon^c = \left\{ u : \|u\| < \varepsilon \text{ or } \|A_n u\| > \frac{1}{\varepsilon} \right\} \quad (6.17)$$

Since

$$\mathbb{E}\|A_n u\| \leq \text{const}, \quad (6.18)$$

then using the Chebyshev inequality, we find

$$\mu \left( \left\{ u : \|Au\| > \frac{1}{\varepsilon} \right\} \right) \leq \text{const } \varepsilon. \quad (6.19)$$

By Theorem 6.2, we have that

$$\mu(\{u : \|u\| < \varepsilon\}) \leq C\varepsilon. \quad (6.20)$$

Finally we write

$$\begin{aligned} \mu(\{u : \tilde{E}_n(u) \in \Gamma\}) &\leq \mu(\{u : \tilde{E}_n(u) \in \Gamma\} \cap A_\varepsilon) + \mu(A_\varepsilon^c) \\ &\leq \frac{C_1}{\alpha(\varepsilon)} \lambda(\Gamma) + C_2 \varepsilon. \end{aligned}$$

That finishes the proof.  $\square$

## 6.2 About the dimension of the measure $\mu$

This subsection is inspired by [Kuk08, KS12] where it was proved that the invariant measures constructed for the Euler equation are not concentrated on a countable union of finite-dimensional compact sets. The proof relies on the Krylov estimate (see section A.9 of [KS12]) for Ito processes. This estimate provides roughly an inequality of the type (6.8) for multi-dimensional processes. In our context the independence needed to use the Krylov estimate leads to solving nonlinear differential equation with order increasing with the size of the underlying vector (process). This is due to the structure of the BO conservation laws, while in the Euler case the components of this vector can be chosen to satisfy this independence. We avoid the resolution of the above-mentioned equation in the 2D case splitting suitably the phase space.

**Theorem 6.3.** *The measure  $\mu$  is of at least two dimensional nature in the sense that any compact set of Hausdorff dimension smaller than 2 has  $\mu$ -measure 0.*

Before proving Theorem 6.3, we describe the general framework.

We use the following splitting of  $H^2(\mathbb{T})$ :

$$H^2(\mathbb{T}) = O \cup O^c, \quad (6.21)$$

where

$$O := \left\{ u : \int u^2 H \partial_x^2 u = 0 \right\}. \quad (6.22)$$

Consider the functionals on  $H^1(\mathbb{T})$  defined by

$$F_j(u) = \frac{1}{j+1} \int u^{j+1}, \quad j = 1, 2. \quad (6.23)$$

Remark that  $F_1$  is preserved by BO and a direct computation shows that for a solution of the BO (1.1) belonging to  $O$ , we have

$$\partial_t F_2(u) = 0. \quad (6.24)$$

Therefore the vector  $F(u) = (F_1(u), F_2(u))$  is constant for any solution  $u$  of BO (1.1) belonging to  $O$ .

On the other hand, consider the following BO conservation laws

$$\begin{aligned} E_0(u) &= \int u^2 \\ E_{1/2}(u) &= \int uH\partial_x u + \frac{1}{3} \int u^3. \end{aligned}$$

Set the following preserved vector

$$E(u) = (E_0(u), E_{1/2}(u)). \quad (6.25)$$

$E(u)$  is in particular constant on  $O^c$  for the solutions of (1.1).

Let  $\mu_1$  and  $\mu_2$  be two measures. We write  $\mu_1 \triangleleft \mu_2$  if there is a continuous increasing function  $f$  vanishing at 0 such that

$$\mu_1(\cdot) \leq f(\mu_2(\cdot)).$$

This implies the absolute continuity of  $\mu_1$  w.r.t.  $\mu_2$ . For  $\nu$  a probability measure on  $H^2$ , we define

$$\nu^O(\cdot) = \nu(\cdot \cap O), \quad (6.26)$$

$$\nu^{O^c}(\cdot) = \nu(\cdot \cap O^c), \quad (6.27)$$

where  $O$  is the set described before.

**Proposition 6.4.** *Suppose  $\lambda_m \neq 0$  for all  $m \in \mathbb{Z}_0$ , then*

1.

$$F_*\mu_\alpha^O \triangleleft l_2, \quad (6.28)$$

2.

$$E_*\mu_\alpha^{O^c} \triangleleft l_2. \quad (6.29)$$

The functions describing the absolute continuity do not depend on  $\alpha$  and  $l_2$  is the Lebesgue measure on  $\mathbb{R}^2$ .

*Proof of Theorem 6.3.* Let  $W$  be an open set of  $H^2$ . Clearly

$$W = (W \cap O) \cup (W \setminus O). \quad (6.30)$$

For  $\varepsilon > 0$ , introduce the set

$$W_\varepsilon = \{u \in W : \text{dist}(u, O) \geq \varepsilon\}, \quad (6.31)$$

We have by regularity of  $\mu_\alpha$  that

$$\mu_\alpha(W \setminus O) = \sup_{\varepsilon > 0} \mu_\alpha(W_\varepsilon). \quad (6.32)$$

Then

$$\mu_\alpha(W) = \mu_\alpha(W \cap O) + \sup_{\varepsilon > 0} \mu_\alpha(W_\varepsilon),$$



and by Proposition 6.4,

$$\mu_\alpha(W) \leq f(l_2(F(W \cap O))) + g(l_2(E(W \setminus O))), \quad (6.33)$$

taking into account the regularity of  $l_2$ . Here  $f$  and  $g$  are the functions describing the absolute continuity established in Proposition 6.4.

Using the Portmanteau theorem, we get

$$\mu(W) \leq f(l_2(F(W \cap O))) + g(l_2(E(W \setminus O))). \quad (6.34)$$

By regularity of  $\mu$  and  $l_2$  the estimate (6.34) holds for any bounded Borelian set  $W$ .

When  $W$  is a compact set of Hausdorff dimension  $\mathcal{H}(W) < 2$ . It is clear that  $E$  and  $F$  are Lipschitz on any compact. Since the Lipschitz maps reduce the Hausdorff dimension, we have the right hand side of (6.34) equals to zero, then so is the left hand one.  $\square$

*Proof of Proposition 6.4.* The prove consists of two steps:

**1. Absolute continuity uniform in  $\alpha$  of  $F$  on the set  $O$ :** The first and second derivative of the functionals  $E_j(u)$  are

$$\begin{aligned} F_j'(u, v) &= \int u^j v, \\ F_j''(u, v) &= j \int u^{j-1} v^2. \end{aligned}$$

Then applying the Ito formula to  $F_j$ , we find

$$F_j(u) = F_j(u(0)) + \int_0^t A_j(s) ds + \sqrt{\alpha} \sum_{m \in \mathbb{Z}_0} \lambda_m \int_0^t (u^j, e_m) d\beta_m(s), \quad j = 1, 2. \quad (6.35)$$

where

$$A_j = -(u^j, H \partial_x^2 u - \alpha \partial_x^2 u) + j \frac{\alpha}{2} \sum_{m \in \mathbb{Z}_0} \lambda_m^2 (u^{j-1}, e_m^2). \quad (6.36)$$

On the set  $O$ , we have

$$\mathbb{E}|A_j| \leq \alpha \text{Const}, \quad (6.37)$$

where Const does not depend on  $\alpha$ .

We consider the  $2 \times 2$ -matrix  $\sigma(u)$ ,  $u \in O$  with entries

$$\sigma_{k,l}(u) = \sum_{m \in \mathbb{Z}_0} \lambda_m^2 (u^k, e_m)(u^l, e_m), \quad k, l = 1, 2. \quad (6.38)$$

It is clear that  $\sigma$  is non-negative. It follows from the Krylov estimate with use the function  $\mathbb{1}_\Gamma$ ,  $\Gamma$  being a Borel set of  $\mathbb{R}^2$ ,

$$\mathbb{E}[(\det(\sigma(u))) \mathbb{1}_\Gamma(F)] \leq C l_2(\Gamma), \quad (6.39)$$

$l_2$  is the Lebesgue measure on  $\mathbb{R}^2$  and  $C$  does not depend on  $\alpha$ .

Define the map

$$\begin{aligned} D : H^1(\mathbb{T}) &\longrightarrow \mathbb{R}_+ \\ u &\longmapsto \det(\sigma(u)) \end{aligned} \quad (6.40)$$

We remark that  $D$  is continuous as composition of continuous maps. We have the following

**Lemma 6.5.** *Suppose  $\lambda_m \neq 0$  for all  $m \in \mathbb{Z}_0$ , then*

$$D(u) = 0 \Rightarrow u \equiv 0. \quad (6.41)$$

*Proof.* Let  $\gamma = (\gamma_1, \gamma_2)$  be a nonzero vector of  $\mathbb{R}^2$ . Then

$$\gamma \sigma(u) \gamma^T = 0 \Rightarrow \sum_{m \in \mathbb{Z}_0} \lambda_m^2 \left( \sum_{j=1}^2 \gamma_j (u^j, e_m) \right)^2 = 0.$$

Since  $\lambda_m \neq 0$  for all  $m$ , we infer that

$$\sum_{j=1}^2 \gamma_j u^j \equiv \text{Const}, \quad (6.42)$$

which is possible only if  $u \equiv 0$ , taking into account that  $\int u = 0$ .  $\square$

Now define

$$A_\varepsilon = \left\{ \|u\|_1^2 \geq \varepsilon, \|u\|_2^2 \leq \frac{1}{\varepsilon} \right\} \subset H^2(\mathbb{T}). \quad (6.43)$$

$A_\varepsilon \cap O$  is a compact in  $H^1(\mathbb{T})$  not containing zero, then by continuity of  $D$ ,  $D(A_\varepsilon \cap O)$  is a compact set in  $\mathbb{R}_+$  not containing 0. Then there is  $c_\varepsilon > 0$  such that  $D(u) \geq c_\varepsilon$  for all  $u \in A_\varepsilon \cap O$ . Using the same splitting argument as in the proof of Theorem 6.1, we get the claimed result.

**2. Absolute continuity uniform in  $\alpha$  of  $E$  on the set  $O^c$ :** We follow the construction above to set a  $2 \times 2$ -matrix  $M$  with entries

$$M_{k,l}(u) = \sum_{m \in \mathbb{Z}_0} \lambda_m^2 E'_0(u, e_m) E'_{1/2}(u, e_m), \quad k, l = 1, 2. \quad (6.44)$$

We have by the Krylov estimate,

$$\mathbb{E}[(\det(M(u))) \mathbb{1}_\Gamma(E)] \leq C l_2(\Gamma), \quad (6.45)$$

where  $C$  does not depend on  $\alpha$  thanks to the preservation of  $E_0$  and  $E_{1/2}$  by the BO flow.

Now  $\det M(u) = 0$  only if there is  $\gamma \in \mathbb{R}$  such that

$$\gamma u + 2H\partial_x u + u^2 \equiv \text{Const}. \quad (6.46)$$

Then

$$\gamma \partial_x u + 2H\partial_x^2 u + 2u\partial_x u \equiv 0. \quad (6.47)$$

Therefore for any  $p > 0$ ,

$$\int u^p H \partial_x^2 u = 0, \quad (6.48)$$

and in particular  $u$  belongs to the set  $O$ , then on  $O^c$   $\det(M(u)) \neq 0$ . We can follow the same splitting argument with use  $A_\varepsilon$  define in the first part to get the result.  $\square$

### 6.3 A Gaussian decay property for the measure $\mu$

We recall that the stochastic force considered in this paper is

$$\eta(t, x) = \frac{d}{dt} \sum_{m \in \mathbb{Z}_0} \lambda_m e_m(x) \beta_m(t),$$

and

$$A_0 = \sum_{m \in \mathbb{Z}_0} \lambda_m^2.$$

In this section, we assume that

$$\lambda_m \leq \frac{c}{m^{\frac{1}{2}+}}, \quad (6.49)$$

where  $c$  is so that,

$$A_0 \leq \frac{1}{2e}. \quad (6.50)$$

**Proposition 6.6.** *In the above setting, the measure  $\mu$  satisfies*

$$\mathbb{E} e^{\|u\|^2} < \infty. \quad (6.51)$$

*In particular, for any  $r > 0$*

$$\mu(\{u \in H : \|u\| > r\}) \leq C e^{-r^2}, \quad (6.52)$$

*where the constant  $C$  does not depend on  $r$ .*

*Proof.* Recall the estimate (5.9):

$$\mathbb{E} \|u\|^{2p} \leq p^p A_0^p,$$

combining this with (6.50), we get

$$\mathbb{E} \|u\|^{2p} \leq \frac{p^p}{2^p e^p}. \quad (6.53)$$

Now using the Stirling formula, we have

$$\frac{\mathbb{E} \|u\|^{2p}}{p!} \leq \frac{p^p}{p! 2^p e^p} \sim_{p \rightarrow \infty} \frac{1}{2^p \sqrt{2\pi p}}. \quad (6.54)$$

Then the serie

$$\sum_{p \geq 1} \frac{\mathbb{E} \|u\|^{2p}}{p!} \quad (6.55)$$

is convergent, and we are led to (6.51). The other claim is obtained combining (6.51) with the Chebyshev inequality.  $\square$

*Remark 6.7.* We obtain in a same way the result of Proposition 6.6 for the viscous measures uniformly in  $\alpha$ .

## A Proof of Lemma 3.3

Remark first that for a solution  $v$  of the nonlinear equation (3.4), we have

$$\partial_t E_n(v) = E'_n(v, \partial_t v) = \alpha E'_n(v, \partial_x^2 v) + \sqrt{\alpha} E'_n(v, \partial_x(vz)) + \alpha \frac{1}{2} E'_n(v, \partial_x(z^2)), \quad n = 0, 1, 2. \quad (\text{A.1})$$

The  $E_n$  are the first three conservation laws of the BO equation.

**The case  $n = 0$ :**  $E'_0(v, w) = 2 \int vw$ . Applying (A.1), we get

$$\begin{aligned} \partial_t E_0(v) + 2\alpha \|v\|_1^2 &= 2\sqrt{\alpha}(v, \partial_x(vz)) + \alpha(v, \partial_x z^2) \\ &= \sqrt{\alpha}(v^2, \partial_x z) + \alpha(v, \partial_x z^2) \\ &\leq \sqrt{\alpha} \|z\|_{\frac{3+}{2}} \|v\|^2 + c\alpha \|v\| \|z\|_1^2 \\ &\leq \sqrt{\alpha} \|z\|_{\frac{3+}{2}} \|v\|^2 + c\alpha (1 + \|v\|^2) \|z\|_1^2. \end{aligned}$$

Remark that  $\|z(\cdot)\|_{3+/2}$  is bounded uniformly in  $\alpha$  on  $[0, T]$  for almost all realizations and in  $t$  by continuity, then with use of the Gronwall inequality we get

$$\sup_{t \in [0, T]} \|v(t)\|^2 + 2\alpha \int_0^T \|v(t)\|_1^2 dt \leq C(T, \omega, \|v_0\|). \quad (\text{A.2})$$

**The case  $n = 1$ :**

$$E'_1(v, w) = -2(\partial_x^2 v, w) + \underbrace{\frac{3}{2}(vH\partial_x v, w) + \frac{3}{4}(v^2, H\partial_x w) + \frac{1}{2}(v^3, w)}_{R'_1(v, w)}. \quad (\text{A.3})$$

It is already shown that

$$|R'_1(v, \partial_x^2 v)| \leq \varepsilon \|v\|_2^2 + C_\varepsilon \|v\|^c \quad (\text{A.4})$$

Then

$$\alpha E'_1(v, \partial_x^2 v) \leq -(2 - \varepsilon)\alpha \|v\|_2^2 + C_\varepsilon \alpha \|v\|^c. \quad (\text{A.5})$$

Taking into account some properties of  $H$ , we can suppose  $R'_1(v, w) = (vH\partial_x v, w) + (v^3, w)$  for our purpose.

Now

$$\begin{aligned} \sqrt{\alpha} |(\partial_x^2 v, \partial_x(vz))| &\leq C\sqrt{\alpha} \|v\|_2 \|v\|_1 \|z\| \\ &\leq \varepsilon \alpha \|v\|_2^2 + C_\varepsilon \|v\|_1^2 \|z\|_1^2 \leq \varepsilon \alpha \|v\|_2^2 + C_{T, \varepsilon, \omega} \|v\|_1^2, \\ \sqrt{\alpha} |(vH\partial_x v, \partial_x(vz))| &= \sqrt{\alpha} |(\partial_x(vH\partial_x v, vz))| \\ &\leq C\sqrt{\alpha} \|v\|_1 \|v\|_2 \|v\| \|z\|_{\frac{1+}{2}} \\ &\leq \varepsilon \alpha \|v\|_2^2 + C_\varepsilon \|v\|_1^2 \|v\|^2 \|z\|_{\frac{1+}{2}}^2 \leq \varepsilon \alpha \|v\|_2^2 + C_{T, \varepsilon, \omega} \|v\|_1^2, \\ \sqrt{\alpha} |(v^3, \partial_x(vz))| &\leq C\sqrt{\alpha} \|v\|_{L^6}^3 \|v\|_1 \|z\|_1 \\ &\leq C\sqrt{\alpha} \|v\|_{1/3}^3 \|v\|_1 \|z\|_1 \\ &\leq C\sqrt{\alpha} \|v\|^2 \|v\|_1^2 \|z\|_1 \leq \sqrt{\alpha} C_{T, \omega} \|v\|_1^2. \end{aligned}$$

To summarise, we have

$$\sqrt{\alpha}E_1'(v, \partial_x(vz)) \leq \varepsilon \|v\|_2^2 + C_{T,\varepsilon,\omega} \|v\|_1^2. \quad (\text{A.6})$$

To estimate the last term, we compute

$$\begin{aligned} \alpha |(\partial_x^2 v, \partial_x z^2)| &\leq C\alpha \|v\|_2 \|z\|_1^2 \\ &\leq \varepsilon \alpha \|v\|_2^2 + \alpha C_\varepsilon \|z\|_1^4, \\ \alpha |(vH\partial_x v, dxz^2)| &\leq C\alpha \|v\|_2 \|v\|_1 \|z\|_1^2/4 \\ &\leq \varepsilon \alpha \|v\|_2^2 + \alpha C_{T,\varepsilon,\omega} \|v\|_1^2, \\ \alpha |(v^3, \partial_x z^2)| &\leq C\alpha \|v\|^2 \|v\|_1 \|z\|_1^2 \\ &\leq \varepsilon \alpha \|v\|_2^2 + C_{T,\varepsilon,\omega}^1 \alpha \|v\|_1^2 + \alpha C_{T,\varepsilon,\omega}^2. \end{aligned}$$

To conclude, we can choose  $\varepsilon$  so that

$$E_1(v) + \alpha \int_0^t \|v(r)\|_2^2 dr \leq E_1(v_0) + C_{T,\omega}^1 \int_0^t \|v(r)\|_1^2 dr + C_{T,\omega}^2 t. \quad (\text{A.7})$$

Recalling the inequality (2.4) and the fact that  $\|v\|$  is bounded uniformly in  $\alpha$ , we have for some  $d > 0$ ,

$$\|v\|_1^2 + 2\alpha \int_0^t \|v(r)\|_2^2 dr \leq E_1(v_0) + C_{T,\omega}^0 + C_{T,\omega}^1 \int_0^t \|v(r)\|_1^2 dr + C_{T,\omega}^2 t. \quad (\text{A.8})$$

With use the Gronwall lemma, we arrive at

$$\sup_{t \in [0,T]} \|v(t)\|_1^2 + 2\alpha \int_0^T \|v(t)\|_2^2 dt \leq C_{T,\omega}(\|v_0\|_1). \quad (\text{A.9})$$

**The case  $n = 2$ :** The form of  $E_2(v)$  (see section 2) combined with some properties of  $H$  allows us to consider, for our purpose, that

$$E_2(v) = \|v\|_2^2 + \int (\partial_x v)^3 + (\partial_x^2 v, H\partial_x v) + (v^2, (\partial_x v)^2) + (v^4, H\partial_x v) + \int v^6. \quad (\text{A.10})$$

Then

$$E_2'(v, w) = 2(\partial_x^2 v, \partial_x^2 w) + 3((\partial_x v)^2, \partial_x w) + 2(\partial_x^2 v, H\partial_x w) + 2(vw, (\partial_x v)^2) \quad (\text{A.11})$$

$$+ 2(v^2 \partial_x v, \partial_x w) + 4(v^3 H\partial_x v, w) + (v^4, H\partial_x w) + 6(v^5, w) \quad (\text{A.12})$$

$$= 2(\partial_x^2 v, \partial_x^2 w) + R_2'(v, w). \quad (\text{A.13})$$

It is already shown in the proof of (2.20) that

$$|R_2'(v, \partial_x^2 v)| \leq \varepsilon \|v\|_3^2 + C_\varepsilon \|v\|^c, \quad (\text{A.14})$$

for some constant  $c > 0$ . Now we have

$$2\alpha(\partial_x^2 v, \partial_x^2(\partial_x^2 v)) = -2\alpha \|v\|_3^2.$$

Then

$$\alpha E_2'(v, \partial_x^2 v) \leq -(2 - \varepsilon)\alpha \|v\|_3^2 + \alpha C_\varepsilon \|v\|^c. \quad (\text{A.15})$$

Now

$$\begin{aligned}
\sqrt{\alpha} |(\partial_x^2 v, \partial_x^2 (\partial_x(vz)))| &\leq C\sqrt{\alpha} \|v\|_3 \|v\|_2 \|z\|_2 \leq \varepsilon \alpha \|v\|_3^2 + C_{T,\varepsilon,\omega} \|v\|_2^2, \\
\sqrt{\alpha} |((\partial_x v)^2, \partial_x^2(vz))| &\leq C_{T,\omega} \sqrt{\alpha} \|v\|_{5/4}^2 \|v\|_2 \\
&\leq C_{T,\omega} \sqrt{\alpha} \|v\|_{1/4} \|v\|_2^2 \leq C_{T,\omega} \sqrt{\alpha} \|v\|_2^2, \\
\sqrt{\alpha} |(\partial_x^2(vz), H\partial_x^2 v)| &\leq \sqrt{\alpha} C \|v\|_2^2 \|z\|_2 \leq \sqrt{\alpha} C_{T,\omega} \|v\|_2^2, \\
\sqrt{\alpha} |(v(\partial_x v)^2, \partial_x^2(vz))| &\leq C\sqrt{\alpha} \|v\| \|z\|_{1+1/2} \|v\|_2^2 \leq C_{T,\omega} \sqrt{\alpha} \|v\|_2^2, \\
\sqrt{\alpha} |(v^2 \partial_x u, \partial_x^2(vz))| &\leq \sqrt{\alpha} C \|v\|_1^3 \|v\|_2 \|z\|_2 \leq C_{T,\omega} \sqrt{\alpha} \|v\|_2^2, \\
\sqrt{\alpha} |(v^3 H\partial_x v, \partial_x^2(vz))| &\leq C\sqrt{\alpha} \|v\|_1^3 \|v\|_2 \|z\|_{1+1/2} \|v\| \leq \sqrt{\alpha} C_{T,\omega} \|v\|_2^2, \\
\sqrt{\alpha} |(v^4, H\partial_x^2(vz))| &\leq \sqrt{\alpha} C \|v\|_1^5 \|z\|_1 \leq \sqrt{\alpha} C_{T,\omega}, \\
\sqrt{\alpha} |(v^5, \partial_x(vz))| &\leq C\sqrt{\alpha} \|v\|_1^5 \|v\| \|z\|_{1+1/2} \leq \sqrt{\alpha} C_{T,\omega}.
\end{aligned}$$

The estimates concerning the term  $\partial_x^2 z^2$  are easier because they do not contain  $v$ . Finally, using the same argument as before (in the case of  $E_1(v)$ ), we arrive at the claimed result.

## B The periodic Hilbert transform

We present in this section a definition of the Hilbert transform in the periodic setting and establish some of its elementary properties. Recall that the sequence defined by

$$e_n(x) = \begin{cases} \frac{\sin(nx)}{\sqrt{\pi}} & \text{if } n < 0, \\ \frac{\cos(nx)}{\sqrt{\pi}} & \text{if } n > 0, \end{cases}$$

forms a basis of  $\dot{H}(\mathbb{T})$ , let us denote this basis by  $\mathcal{B}$ . We define the Hilbert transform on  $\mathcal{B}$  by

$$He_n(x) = \operatorname{sgn}(n)e_{-n}(x), \quad (\text{B.1})$$

where

$$\operatorname{sgn}(p) = \begin{cases} 1 & \text{if } p > 0, \\ 0 & \text{if } p = 0, \\ -1 & \text{if } p < 0. \end{cases}$$

We first remark that  $H$  defines an isometry on  $\dot{H}$ .

**Proposition B.1.** *Let  $f, g \in \dot{H}(\mathbb{T})$ , then*

$$H^2 f = -f \quad (\text{B.2})$$

$$\int_{\mathbb{T}} Hf = 0 \quad (\text{B.3})$$

$$(g, Hf) = -(Hg, f) \quad (\text{B.4})$$

$$\widehat{Hf}_0(p) = -i \operatorname{sgn}(p) \hat{f}_0(p), \quad (\text{B.5})$$

where  $\hat{h}_0$  denotes the complex Fourier coefficient of a function  $h$ . We define it below.

Define now the Fourier coefficients associated to a function  $f$  in  $\dot{H}$ :

$$\hat{f}_1(n) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{T}} \cos(nx) f(x) dx \quad (\text{B.6})$$

$$\hat{f}_2(n) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{T}} \sin(nx) f(x) dx. \quad (\text{B.7})$$

The function  $f$  is represented in  $\mathcal{B}$  as follow

$$f(x) = \sum_{n>0} (\hat{f}_1(n) e_n(x) - \hat{f}_2(n) e_{-n}(x)). \quad (\text{B.8})$$

Hence the Hilbert transform of  $f$  can be expressed as

$$Hf(x) = \sum_{n>0} (\hat{f}_1(n) e_{-n}(x) + \hat{f}_2(n) e_n(x)). \quad (\text{B.9})$$

The complex Fourier coefficient is defined by

$$\hat{f}_0(p) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-ipx} f(x) dx. \quad (\text{B.10})$$

The relation between the three Fourier coefficients of  $f$  is

$$\hat{f}_0(p) = \frac{\hat{f}_1(p) - i \operatorname{sgn}(p) \hat{f}_2(p)}{2}. \quad (\text{B.11})$$

*Proof of Proposition B.1.* (B.3) follows immediately from (B.9). Now from (B.8) and (B.9), we can easily deduce that

$$H^2 f(x) = - \sum_{n>0} (\hat{f}_1(n) e_n(x) - \hat{f}_2(n) e_{-n}(x)) = -f(x). \quad (\text{B.12})$$

and (B.2) is showed.

From (B.9), we infer that

$$\widehat{Hf}_1(p) = -\hat{f}_2(p), \quad \widehat{Hf}_2(p) = \hat{f}_1(p). \quad (\text{B.13})$$

Thus using the relation (B.11), we write

$$\begin{aligned} \widehat{Hf}_0(p) &= \frac{-\hat{f}_2(p) - i \operatorname{sgn}(p) \hat{f}_1(p)}{2} \\ &= \frac{-i \operatorname{sgn}(p) (\hat{f}_1(p) - i \operatorname{sgn}(p) \hat{f}_2(p))}{2} \\ &= -i \operatorname{sgn}(p) \hat{f}_0(p), \end{aligned}$$

and we arrived at (B.5).

To prove (B.4), we compute

$$\begin{aligned} (g, Hf) &= \sum_{n>0} \hat{f}_1(n) \int_{\mathbb{T}} g(x) e_{-n}(x) dx + \sum_{n>0} \hat{f}_2(n) \int_{\mathbb{T}} g(x) e_n(x) dx \\ &= - \sum_{n>0} \hat{f}_1(n) \hat{g}_2(n) + \sum_{n>0} \hat{g}_1(n) \hat{f}_2(n) \\ &= - \sum_{n>0} \hat{g}_2(n) \int_{\mathbb{T}} f(x) e_n(x) dx - \sum_{n>0} \hat{g}_1(n) \int_{\mathbb{T}} f(x) e_{-n}(x) dx \\ &= - \int_{\mathbb{T}} f(x) \sum_{n>0} (\hat{g}_1(n) e_{-n}(x) + \hat{g}_2(n) e_n(x)) \\ &= -(Hg, f). \end{aligned}$$

□

**Acknowledgements.** I thank my thesis supervisors Armen Shirikyan and Nikolay Tzvetkov for many fruitful discussions. This research is supported by "Région Ile-de-France" in the program DIM-RDMath.

## References

- [ABFS89] L. Abdelouhab, J. L. Bona, M. Felland, and J.C. Saut. Nonlocal models for nonlinear, dispersive waves. *Physica D: Nonlinear Phenomena*, 40(3):360–392, 1989.
- [Ben67] T. B. Benjamin. Internal waves of permanent form in fluids of great depth. *Journal of Fluid Mechanics*, 29(03):559–592, 1967.
- [BP08] N. Burq and F. Planchon. On well-posedness for the Benjamin-Ono equation. *Math. Ann.*, 340(3):497–542, 2008.
- [Den15] Y. Deng. Invariance of the Gibbs measure for the Benjamin-Ono equation. *J. Eur. Math. Soc. (JEMS)*, 2015.
- [DPZ14] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*. Cambridge University Press, Cambridge, 2014.
- [DTV15] Y. Deng, N. Tzvetkov, and N. Visciglia. Invariant measures and long time behaviour for the Benjamin-Ono equation III. *Comm. Math. Phys.*, 339(3):815–857, 2015.
- [IK07] A. Ionescu and C. Kenig. Global well-posedness of the Benjamin-Ono equation in low-regularity spaces. *Journal of the American Mathematical Society*, 20(3):753–798, 2007.
- [IK09] A. Ionescu and C. Kenig. Local and global well-posedness of periodic KP-I equations. *Mathematical Aspects of Nonlinear Dispersive Equations. Ann. Math. Stud.*, 163:181–211, 2009.
- [Kin09] F. W. King. *Hilbert transforms*. Cambridge University Press, Cambridge, 2009.
- [KS91] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*. Graduate texts in Mathematics, Springer-Verlag, New York, 1991.
- [KS04] S. Kuksin and A. Shirikyan. Randomly forced CGL equation: stationary measures and the inviscid limit. *Journal of Physics. A. Mathematical and General*, 37:3805–3822, 2004.
- [KS12] S. Kuksin and A. Shirikyan. *Mathematics of Two-Dimensional Turbulence*. Cambridge University Press, Cambridge, 2012.
- [KT03] H. Koch and N. Tzvetkov. On the local well-posedness of the Benjamin-Ono equation in  $H^s(\mathbb{R})$ . *International Mathematics Research Notices*, 2003(26):1449–1464, 2003.
- [Kuk04] S. Kuksin. The Eulerian limit for 2D statistical hydrodynamics. *J. Statist. Phys.*, 115(1-2):469–492, 2004.



- [Kuk08] S. Kuksin. On distribution of energy and vorticity for solutions of 2d Navier-Stokes equation with small viscosity. *Communications in Mathematical Physics*, 284(2):407–424, 2008.
- [Mat84] Y. Matsuno. *Bilinear transformation method*. Mathematics in Science and Engineering. Academic Press, Inc., Orlando, FL, 1984.
- [Mol08] L. Molinet. Global well-posedness in  $L^2$  for the periodic Benjamin-Ono equation. *American Journal of Mathematics*, 130(3):635–683, 2008.
- [MP12] L. Molinet and D. Pilod. The Cauchy problem for the Benjamin-Ono equation in  $L^2$  revisited. *Analysis & PDE*, 5(2):365–395, 2012.
- [MST01] L. Molinet, J. C. Saut, and N. Tzvetkov. Ill-posedness issues for the Benjamin-Ono and related equations. *SIAM journal on mathematical analysis*, 33(4):982–988, 2001.
- [Shi11] A. Shirikyan. Local times for solutions of the complex Ginzburg-Landau equation and the inviscid limit. *J. Math. Anal. Appl.*, 384(1):130–137, 2011.
- [Sy16] M. Sy. Invariant measure and large time dynamics for the Klein-Gordon equation in 3D. *In preparation*, 2016.
- [Tao04] T. Tao. Global well-posedness of the Benjamin-Ono equation in  $H^1(\mathbb{R})$ . *Journal of Hyperbolic Differential Equations*, 1(01):27–49, 2004.
- [TV13] N. Tzvetkov and N. Visciglia. Gaussian measures associated to the higher order conservation laws of the Benjamin-Ono equation. *Ann. Sci. Éc. Norm. Supér. (4)*, 46(2):249–299, 2013.
- [TV14] N. Tzvetkov and N. Visciglia. Invariant measures and long-time behavior for the Benjamin-Ono equation. *Int. Math. Res. Not. IMRN*, (17):4679–4714, 2014.
- [TV15] N. Tzvetkov and N. Visciglia. Invariant measures and long time behaviour for the Benjamin-Ono equation II. *J. Math. Pures Appl. (9)*, 103(1):102–141, 2015.
- [Tzv06] N. Tzvetkov. Ill-posedness issues for nonlinear dispersive equations. In *Lectures on nonlinear dispersive equations*, volume 27 of *Gakuto Internat. Ser. Math. Sci. Appl.*, pages 63–103. 2006.
- [Tzv10] N. Tzvetkov. Construction of a Gibbs measure associated to the periodic Benjamin-Ono equation. *Probability theory and related fields*, 146(3-4):481–514, 2010.
- [Zhi01a] P. E. Zhidkov. *Korteweg-de Vries and nonlinear Schrödinger equations: qualitative theory*. Number 1756. Springer Science & Business Media, 2001.
- [Zhi01b] P. E. Zhidkov. On an infinite sequence of invariant measures for the cubic nonlinear Schrödinger equation. *Int. J. Math. Math. Sci.*, 28(7):375–394, 2001.